Metric Spaces: General properties

Inner Product $\Rightarrow$ Norm $\Rightarrow$ Metric

I Metric Space:

Metric Space: pair (X,d) where X is a nonempty set, and $d: X\times X \longrightarrow [0,\infty)$ a function called metric.

(1) $d(x,y) \geq 0$
(2) $d(x,y) = 0 \iff x=y$
(3) $d(x,y) = d(y,x)$
(4) $d(x,y) \leq d(x,z) + d(z,y)$ $\longrightarrow$ Triangle ineq

Hold $\forall x,y,z \in X$.

Note: (4) $\Rightarrow$ $d(x_1,x_n) \leq d(x_1,x_2) + d(x_2,x_3) + \cdots + d(x_{n-1},x_n)$ $\forall x_1,\dots,x_n \in X$.

Examples:

① $X=\mathbb{R}$, $d(x,y) = |x-y|$

② $X=\mathbb{R}^n$, $d(x,y) = \left[\sum_{k=1}^{n} (x_k-y_k)^2\right]^{1/2}$ where $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n) \in \mathbb{R}^n$

③ $X=\mathbb{R}^n$, $d(x,y) = \max_{1\leq k\leq n} |x_k-y_k|$ $\longrightarrow$ "Box metric"

④ $X = C([a,b])$, $d(f,g) = \max_{x\in[a,b]} |f(x)-g(x)|$

$\hookrightarrow$ space of all cont f's $f:[a,b]\to\mathbb{R}$

$(f+g)(x) = f(x)+g(x)$, $(\alpha f)(x) = \alpha f(x)$, f,g cont $\Rightarrow$ $|f(x)-g(x)|$ has a $\overset{\text{finite}}{\text{max}}$ on $[a,b]$

⑤ $X = C([a,b])$, $d(f,g) = \left(\int_a^b |f(x)-g(x)|^2 dx\right)^{1/2}$

⑥ $X\neq\emptyset$: $d(x,y) = \begin{cases} 1, & x\neq y \\ 0, & x=y \end{cases}$ $\longrightarrow$ discrete metric

Note: Ex 1-5 are induced by a norm, in Ex 2, 5 the norm is induced by an Inner Product.

II Normed Space:

Norm: X a Vector Space over $\mathbb{R}$; a function $\|\cdot\|: X \longrightarrow [0,\infty)$ is called a norm if:

(1) $\|x\| \geq 0$
(2) $\|x\| = 0 \iff x = 0_X$ ($\hookrightarrow$ zero vector in X)
(3) $\|\alpha x\| = |\alpha|\cdot\|x\|$
(4) $\|x+y\| \leq \|x\| + \|y\|$

$\forall x,y\in X$
$\forall \alpha \in \mathbb{R}$.

Normed Space: pair $(X,\|\cdot\|)$ where X is a Vspace over $\mathbb{R}$, $\|\cdot\|$ a norm on X.

Prop: $(X,\|\cdot\|)$ a normed space $\Longrightarrow$ $\boxed{d(x,y) := \|x-y\|}$ is a metric on X $\rightsquigarrow$ "metric induced by a norm"

(PF) • $d(x,y) = \|x-y\| = \|(-1)(y-x)\| \overset{(3)}{=} |-1|\cdot\|y-x\| = \|y-x\| = d(y,x)$

• $d(x,y) = \|x-y\| = \|x-z+z-y\| \overset{(4)}{\leq} \|x-z\| + \|z-y\| = d(x,z)+d(z,y)$

• $d(x,y) = \|x-y\| \geq 0$

• $d(x,y) = 0 \iff \|x-y\| = 0 \iff x-y=0 \iff x=y$.

Examples:

① $X=\mathbb{R}$, $\|x\| = |x|$

② $X=\mathbb{R}^n$, $\|x\| = \left(\sum_{k=1}^{n} x_k^2\right)^{1/2}$ $\rightsquigarrow$ induced by $\langle x,y\rangle = \sum_{k=1}^{n} x_k y_k$

③ $X=\mathbb{R}^n$, $\|x\| = \max_{1\leq k\leq n} |x_k|$

$\|x+y\| = \max_{1\leq k\leq n} |x_k+y_k| = |x_{k^*}+y_{k^*}| \leq |x_{k^*}|+|y_{k^*}| \leq \max_{1\leq k\leq n} |x_k| + \max_{1\leq k\leq n} |y_k| = \|x\|+\|y\|$

④ $X = C([a,b])$, $\|f\| = \max_{x\in[a,b]} |f(x)|$

⑤ $X = C([a,b])$, $\|f\| = \left(\int_a^b |f(x)|^2 dx\right)^{1/2}$ $\rightsquigarrow$ induced by $\langle x,y\rangle = \int_a^b f(x)g(x)dx$

⑥ $X\neq\emptyset$ Vspace, d the discrete metric on X: d is <u>NOT</u> induced by a norm

($\because$) Suppose it is: $d(x,y) = \|x-y\| = \begin{cases} 1, & x\neq y \\ 0, & x=y \end{cases}$: $\forall \alpha\in\mathbb{R}$, $1 = d(\alpha x,\alpha y) = \|\alpha x-\alpha y\| = |\alpha|\cdot\|x-y\| = |\alpha|\cdot d(x,y) = |\alpha| \Rightarrow |\alpha| = 1$ $\forall \alpha\in\mathbb{R}$ $\Rightarrow\Leftarrow$

Note: Norms induced by inner product have an Euclidean geometry.

III Inner Product Spaces :

Inner Product : X a Vspace over $\mathbb{R}$: $\langle\cdot,\cdot\rangle : X\times X \longrightarrow \mathbb{R}$ is an IP on X if : (1) $\langle x,x\rangle \geq 0$
(2) $\langle x,x\rangle = 0 \Leftrightarrow x = 0_X$ $\quad \forall x,y,z \in X$
(3) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x,y\rangle + \beta\langle x,z\rangle$ $\quad \forall \alpha,\beta \in \mathbb{R}$
(4) $\langle x,y\rangle = \langle y,x\rangle$

Note : $\langle\cdot,\cdot\rangle$ is also linear WRT 1^{st} var : $\langle \alpha y+\beta z, x\rangle = \langle x, \alpha y+\beta z\rangle = \alpha\langle x,y\rangle + \beta\langle x,z\rangle = \alpha\langle y,x\rangle + \beta\langle z,x\rangle$

Complex IP : (4) $\rightsquigarrow \langle x,y\rangle = \overline{\langle y,x\rangle}$, and $\langle\cdot,\cdot\rangle$ is linear WRT $\left\{\begin{matrix}1^{st}\\2^{nd}\end{matrix}\right\}$ Var and antilinear WRT $\left\{\begin{matrix}2^{nd}\\1^{st}\end{matrix}\right\}$ Var. $\left\{\begin{matrix}\text{Pure math}\\\text{Appl math / phys}\end{matrix}\right\}$

Cauchy-Schwartz : X a Vspace over $\mathbb{R}$, $\langle\cdot,\cdot\rangle$ an IP on X : $\boxed{|\langle x,y\rangle| \leq \sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle}}$ $\forall x,y \in X$.

(PF) • if $x = 0_X$ or $y = 0_X$: $|\langle x,y\rangle| = 0 \leq 0 = \sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle}$ Since : $\langle x, 0_X\rangle = \langle x, 0\cdot 0_X\rangle = 0\langle x,0_X\rangle = 0$.
• if $x,y \neq 0_X$: $f(t) = \langle x+ty, x+ty\rangle$, $t\in\mathbb{R}$: $f(t)\geq 0$ $\forall t$ and $f(t) = \langle x,x\rangle + 2t\langle x,y\rangle + t^2\langle y,y\rangle \Rightarrow f'(t) = 2\langle x,y\rangle + 2t\langle y,y\rangle \Rightarrow t_{min} = -\frac{\langle x,y\rangle}{\langle y,y\rangle}$
So $f(t_{min}) = \langle x,x\rangle - 2\frac{\langle x,y\rangle^2}{\langle y,y\rangle} + \frac{\langle x,y\rangle^2}{\langle y,y\rangle} = \langle x,x\rangle - \frac{\langle x,y\rangle^2}{\langle y,y\rangle} \geq 0 \Rightarrow \langle x,x\rangle\langle y,y\rangle \geq \langle x,y\rangle^2 \Rightarrow |\langle x,y\rangle| \leq \sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle}$.

Corr : $(X,\langle\cdot,\cdot\rangle)$ an IP space $\Rightarrow \boxed{\|x\| = \sqrt{\langle x,x\rangle}}$ is a norm on X $\rightsquigarrow$ "Norm induced by an IP"

(PF) $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow \langle x,x\rangle = 0 \Leftrightarrow x = 0_X$, $\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x\rangle} = \sqrt{\alpha^2\langle x,x\rangle} = |\alpha|\sqrt{\langle x,x\rangle} = |\alpha|\cdot\|x\|$
$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle \leq \langle x,x\rangle + 2|\langle x,y\rangle| + \langle y,y\rangle \leq \langle x,x\rangle + 2\sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle} + \langle y,y\rangle = \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2 = (\|x\|+\|y\|)^2$

Angle between Vectors : $(X,\langle\cdot,\cdot\rangle)$ an IP Space : $x,y \in X$ nonzero vectors :

Angle between x and y : $\exists!\, \theta \in [0,\pi]$ ST $\boxed{\cos\theta = \frac{\langle x,y\rangle}{\|x\|\cdot\|y\|}} \Rightarrow \boxed{\langle x,y\rangle = \|x\|\cdot\|y\|\cos\theta}$ $\quad$ (PF) $-1 \leq \frac{\langle x,y\rangle}{\|x\|\cdot\|y\|} \leq 1$ Since Cauchy Schwartz : $|\langle x,y\rangle| \leq \|x\|\cdot\|y\|$

Orthogonality : $x \perp y \Leftrightarrow \langle x,y\rangle = 0$, i.e. $\theta = \frac{\pi}{2}$.

Pythagora's theorem : $(X,\langle\cdot,\cdot\rangle)$ an IP Space : $x\perp y \Leftrightarrow \langle x,y\rangle = 0 \Leftrightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$

(PF) $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2 \Rightarrow \boxed{\langle x,y\rangle = \frac{\|x+y\|^2 - \|x\|^2 - \|y\|^2}{2} = 0}$

Examples : (2) $X = \mathbb{R}^n$, $\langle x,y\rangle = \sum_{k=1}^{n} x_k y_k$; $X = \mathbb{C}$: $\langle x,y\rangle = \sum_{k=1}^{n} \overline{x_k} y_k$
(5) $X = C([a,b])$, $\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$

Ex : Geometry on $C([a,b])$: $X = C([0,2\pi])$, $\sin x, \cos x \in X$: $\langle \sin x, \cos x\rangle = \int_0^{2\pi} \sin x\cos x\,dx = 0 \Rightarrow \sin x \perp \cos x$.

Parallelogram Law . $(X,\langle\cdot,\cdot\rangle)$ an IP space, $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ the induced norm : $\boxed{\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)}$ holds $\forall x,y \in X$

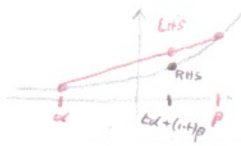
(PF) $\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle = \langle x,x\rangle + \cancel{2\langle x,y\rangle} + \langle y,y\rangle + \langle x,x\rangle - \cancel{2\langle x,y\rangle} + \langle y,y\rangle = 2(\|x\|^2 + \|y\|^2)$

Thm : $(X,\|\cdot\|)$ a normed space : $\exists \langle\cdot,\cdot\rangle$ ST $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle} \Longleftrightarrow \|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ $\forall x,y \in X$ $\rightsquigarrow$ Parallelogram Law

(PF) $\Rightarrow$ already done
$\Leftarrow$ if # law holds : show that $\langle x,y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$ is an IP. $\rightsquigarrow$ 4th tutorial!

Examples : (3) $X = \mathbb{R}^n$, $\|x\| = \max_{1\leq k\leq n} |x_k|$ is NOT induced by an IP for $n\geq 2$! (PF) Try : $\left\{\begin{matrix} x = (1,1,0,\dots,0) \\ y = (1,-1,0,\dots,0)\end{matrix}\right. \Rightarrow \left\{\begin{matrix} x+y = (2,0,0,\dots,0) \\ x-y = (0,2,0,\dots,0)\end{matrix}\right. \Rightarrow \left\{\begin{matrix} \|x+y\|^2 + \|x-y\|^2 = 2^2+2^2 = 8 \\ 2(\|x\|^2 + \|y\|^2) = 2(1^2+1^2) = 4\end{matrix}\right.$
$\Rightarrow$ # law is not verified!
(4) $X = C([a,b])$, $\|f\| = \max_{x\in[a,b]} |f(x)|$ is NOT induced by an IP ! (PF) Try : $[a,b] = [0,1]$: $\left\{\begin{matrix} f(x) = x \\ g(x) = e^x\end{matrix}\right. \Rightarrow \left\{\begin{matrix} f(x)+g(x) = x+e^x \\ f(x)-g(x) = x-e^x\end{matrix}\right. \Rightarrow \left\{\begin{matrix} \|f+g\|^2 + \|f-g\|^2 = (1+e)^2 + 1 \\ 2(\|f\|^2 + \|g\|^2) = 2(1+e^2)\end{matrix}\right.$
$\Rightarrow 1+2e+e^2 \neq 2+2e^2 \rightsquigarrow$ # law breaks

IV Hölder & Minkovski's Inequalities :



Convex functions: f is convex $\iff f(t\alpha+(1-t)\beta) \leq t f(\alpha)+(1-t)f(\beta) \ \forall t \in [0,1]$

Lemma = $a,b \geq 0,\ p,q>1$ ST $\frac{1}{p}+\frac{1}{q}=1$: then $\boxed{ab \leq \frac{1}{p}a^p+\frac{1}{q}b^q}$

(pf) • $a=0$ or $b=0$: $ab=0 \leq \frac{1}{p}a^p+\frac{1}{q}b^q$ as $a\geq 0,\ b\geq 0$

• $a>0, b>0$: let $\begin{cases}\alpha = p\ln a \\ \beta = q\ln b\end{cases} \Rightarrow \begin{cases} a=e^{\alpha/p} \\ b=e^{\beta/q}\end{cases} \Rightarrow ab = e^{\frac{\alpha}{p}+\frac{\beta}{q}} \overset{f(x)=e^x}{=} f(\frac{1}{p}\alpha+\frac{1}{q}\beta) = f(\frac{1}{p}\alpha+(1-\frac{1}{p})\beta) \overset{f(x)=e^x \text{ is convexe}}{\leq} \frac{1}{p}e^{\alpha}+\frac{1}{q}e^{\beta} = \frac{1}{p}a^p+\frac{1}{q}b^q$

Hölder Inequality = $x,y \in \mathbb{R}^n,\ p,q>1$ ST $\frac{1}{p}+\frac{1}{q}=1 \Longrightarrow \boxed{\sum_{k=1}^n |x_k|\cdot|y_k| \leq \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}\cdot\left(\sum_{k=1}^n |y_k|^q\right)^{1/q}}$

Proof: $A=\left(\sum_{k=1}^n |x_k|^p\right)^{1/p}$, $B=\left(\sum_{k=1}^n |y_k|^q\right)^{1/q}$: Goal: $\sum_{k=1}^n |x_k||y_k| = A\cdot B$

• $A=0$ or $B=0 \Rightarrow x=0$ or $y=0 \Rightarrow 0 = AB = 0$

• $\frac{|x_k|}{A}\cdot\frac{|y_k|}{B} \overset{\text{Lemma}}{\leq} \frac{1}{p}\cdot\left(\frac{|x_k|}{A}\right)^p+\frac{1}{q}\left(\frac{|y_k|}{B}\right)^q \Rightarrow \sum_{k=1}^n \frac{|x_k||y_k|}{A\ B} \leq \frac{1}{p}\sum_{k=1}^n\frac{|x_k|^p}{A^p}+\frac{1}{q}\sum_{k=1}^n\frac{|y_k|^q}{B^q} = \frac{1}{p}\cdot 1+\frac{1}{q}\cdot 1 = \frac{1}{p}+\frac{1}{q}=1$

$\sum_{k=1}^n \frac{|x_k|^p}{A^p} = \frac{\sum_{k=1}^n |x_k|^p}{\sum_{k=1}^n |x_k|^p} = 1$

$\Rightarrow \sum_{k=1}^n |x_k|\cdot|y_k| \leq A\cdot B.$

$\sum_{k=1}^n \frac{|y_k|^q}{B^q} = \frac{\sum_{k=1}^n |y_k|^q}{\sum_{k=1}^n |y_k|^q} = 1$

Minkovski Inequality = $x,y\in\mathbb{R}^n,\ p\geq 1 \Rightarrow \boxed{\left(\sum_{k=1}^n (|x_k|+|y_k|)^p\right)^{1/p} \leq \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}+\left(\sum_{k=1}^n |y_k|^p\right)^{1/p}}$

Proof: • $p=1$: Trivial equality

• $p>1$:

$$\sum_{k=1}^n (|x_k|+|y_k|)^p = \sum_{k=1}^n (|x_k|+|y_k|)^{p-1}(|x_k|+|y_k|) = \sum_{k=1}^n (|x_k|+|y_k|)^{p-1}|x_k| + \sum_{k=1}^n (|x_k|+|y_k|)^{p-1}|y_k|$$

$$\overset{\text{(Hölder)}}{\leq} \left(\sum_{k=1}^n (|x_k|+|y_k|)^{q\cdot(p-1)}\right)^{1/q}\cdot\left(\sum_{k=1}^n |x_k|^p\right)^{1/p}+\left(\sum_{k=1}^n (|x_k|+|y_k|)^{q\cdot(p-1)}\right)^{1/q}\cdot\left(\sum_{k=1}^n |y_k|^p\right)^{1/p}$$

$\left(\frac{1}{q}=1-\frac{1}{p}=\frac{p-1}{p} \Rightarrow q(p-1)=p\right)$

$$= \left(\sum_{k=1}^n (|x_k|+|y_k|)^p\right)^{1/q}\cdot\left(\left(\sum_{k=1}^n |x_k|^p\right)^{1/p}+\left(\sum_{k=1}^n |y_k|^p\right)^{1/p}\right)$$

$$\left(\sum_{k=1}^n (|x_k|+|y_k|)^p\right)^{1-\frac{1}{q}} \leq \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}+\left(\sum_{k=1}^n |y_k|^p\right)^{1/p}$$

$\hookrightarrow 1-\frac{1}{q}=\frac{1}{p} \Rightarrow$ Minkovski! ✓

V ℓ^p norms :

L^p norm = $X=\mathbb{R}^n$, $\|x\|_p = \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}$ is a norm on $\mathbb{R}^n$ for $p\geq 1$ <u>only</u>.

(pf) $\|x\|_p \geq 0$, $\|x\|_p=0 \iff X=0_x$, $\|\alpha x\|_p = |\alpha|\cdot\|x\|_p$, $\|x+y\|_p = \left(\sum_{k=1}^n |x_k+y_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^n (|x_k|+|y_k|)^p\right)^{1/p} \underset{\text{Minkovski}}{\leq} \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}+\left(\sum_{k=1}^n |y_k|^p\right)^{1/p} = \|x\|_p+\|y\|_p$

Note: if $0<p<1$, $\|\cdot\|_p$ is <u>NOT</u> a norm : Triangle ineq breaks for $n\geq 2$: $\begin{matrix}(1,0,\dots,0)=x\\(0,1,0,\dots,0)=y\end{matrix} \Rightarrow \begin{matrix}\|x\|_p=1=\|y\|_p\\ \|x+y\|_p=2^{1/p}\end{matrix} \Rightarrow 1+1=2=2^{1/p} \Rightarrow \frac{1}{p}=1 \Rightarrow p=1$ as $0<p<1$.

Prop: • $f: p\longmapsto \|x\|_p$: $\lim_{p\to\infty}\|x\|_p = \max_{1\leq k\leq n}|x_k| = \|x\|_\infty \rightsquigarrow$ box norm! $|x_0| = \max_{1\leq k\leq n}|x_k|$: $|x_0| \leq \left(\sum_{k=1}^n |x_k|^p\right)^{1/p} \leq \left(\sum |x_0|^p\right)^{1/p} = |x_0|\cdot n^{1/p} \rightsquigarrow |x_0| \leq \|x\|_\infty \leq$ $\Rightarrow \|x\|_\infty = |x_0|$

• $d(x,y) = \sum_{k=1}^n |x_k-y_k|^p$, $0<p<1$ is a metric on $\mathbb{R}^n$

$d(x,y)\geq 0$; $d(x,y)=0 \iff |x_k-y_k|=0\ \forall k \iff x_k=y_k\ \forall k \iff x=y$; $d(x,y)=d(y,x)$

$d(x,y) = \sum_{k=1}^n |x_k-y_k|^p \leq \sum_{k=1}^n (|x_k-z_k|+|z_k-y_k|)^p \leq \sum_{k=1}^n (|x_k-z_k|^p+|z_k-y_k|^p)$

So $d(x,y) \leq d(x,z)+d(z,y)$

$(a+b)^p \leq a^p+b^p \quad \forall a,b\geq 0$

(pf) Look at $\phi(x) = a^p+x^p-(a+x)^p \geq \phi(0)=0$

• $\|\cdot\|_p$ is induced by an inner product $\iff p=2$ (and $n\geq 2$)

(pf) • // Low fails for $p\neq 2$: $X=(1,1,0,0,\dots,0)$, $Y=(1,-1,0,\dots,0) \Rightarrow \begin{cases}\|x+y\|_p=2\\ \|x-y\|_p=2\end{cases}$ $\begin{cases}\|x\|_p=2^{1/p}\\ \|y\|_p=2^{1/p}\end{cases}$ $\Rightarrow \begin{cases}\|x+y\|_p^2+\|x-y\|_p^2=8\\ 2(\|x\|_p^2+\|y\|_p^2)=4\cdot 2^{2/p}\end{cases} \Rightarrow 8=4\cdot 2^{2/p} \Rightarrow 2^{2/p}=2^{2/2}$

$\Rightarrow$ breaks $\forall p\neq 2$

• $\underline{p=2}$: $\|x\|_p = \left(\sum_{k=1}^n |x_k|^2\right)^{1/2} \rightsquigarrow$ induced by $\langle x,y\rangle = \sum_{k=1}^n x_k y_k$

VI Basic definitions inside metric spaces

① Metric Subspaces :

<u>Metric Subspaces</u>: Let (X,d) a metric space, $Y \subset X$: the restriction of d to $Y \times Y$ is a metric
$\rightarrow (Y,d)$ is a metric space WRT $(X,d) \Rightarrow (Y,d)$ is a metric subspace of (X,d)

<u>Ex</u>: $X = \mathbb{R}^2$, $Y = \{x^2+y^2=1 : (x,y) \in \mathbb{R}^2\} \subseteq \mathbb{R}^2$: $\|X\| = \sqrt{x_1^2+x_2^2}$

<u>Prop</u>: the <u>finite</u> product of metric subspaces (X_k, d_k), $1 \le k \le n$: $\boxed{\begin{cases} X = \overset{n}{\underset{k=1}{\times}} X_k = \prod_{k=1}^{n} X_k = \{X = (x_1, \dots, x_n) : x_k \in X_k\} \\ d(x,y) = \left[\sum_{k=1}^{n} (d_k(x_k,y_k))^2\right]^{1/2} \end{cases}}$
Then (X,d) is a metric space, d is the product metric

(∵) $x,y,z \in X$: $d(x,y) = \left(\sum_{k=1}^{n} (d_k(x_k,y_k))^2\right)^{1/2} \le \left(\sum_{k=1}^{n} (d_k(x_k,z_k) + d(z_k,y_k))^2\right)^{1/2} \underset{\text{Minkowski}}{\le} \left[\sum_{k=1}^{n} (d_k(x_k,z_k))^2\right]^{1/2} + \left[\sum_{k=1}^{n} (d_k(z_k,y_k))^2\right]^{1/2} = d(x,z) + d(z,y)$.

② Equivalence of metrics :

<u>Equivalent metrics</u>: 2 metrics d_1, d_2 on a set $X \neq \emptyset$ are equivalent if $\exists c_1, c_2 > 0$ ST $\boxed{c_1 d_2(x,y) \le d_1(x,y) \le c_2 d_2(x,y)}$ $\forall x,y \in X$

<u>Prop</u>: $d_1 \sim d_2 \Leftrightarrow d_1$ and d_2 are equivalent metrics on X is a relation of equivalence on the set of all metrics on X.
<u>Check</u>: $d_1 \sim d_2 \Rightarrow d_2 \sim d_1$: $d_1 \sim d_2 \Rightarrow \frac{1}{c_1} d_1(x,y) \le d_2(x,y) \le \frac{1}{c_2} d(x,y) \Rightarrow d_2 \sim d_1$; $d_1 \sim d_1$: $c_1 = c_2 = 1$;
$d_1 \sim d_2, d_2 \sim d_3 \Rightarrow d_1 \sim d_3$: $c_1' c_1 d_3(x,y) \le c_1 d_2(x,y) \le d_1(x,y) \le c_2 d_2(x,y) \le c_2' c_2 d_2(x,y)$, $c_1 c_1', c_2 c_2' > 0$

<u>Equivalent norms</u>: X a vector space, and $\|\cdot\|_1, \|\cdot\|_2$ norms on X : $\|\cdot\|_1 \sim \|\cdot\|_2$ if $\exists c_1, c_2 > 0$ ST $\boxed{c_1 \|x\|_2 \le \|x\|_1 \le c_2 \|x\|_2}$ $\forall x \in X$

<u>Prop</u>: $\dim X < \infty \Rightarrow$ any 2 norms are equivalent. $\rightarrow$ proof: need compactness.
⚠ <u>NOT</u> true for $\dim X = \infty$

<u>Ex</u>: • $X = \mathbb{R}^n$: $\|X\|_1 = \sum_{k=1}^{n} |x_k|$ and $\|x\|_\infty = \max_{1 \le k \le n} |x_k|$ are <u>equivalent</u> : $\|x\|_\infty \le \|x\|_1 \le n \|x\|_\infty$ $\rightarrow$ fixed here! $\mathbb{R}^n$
• $X = \prod_{k=1}^{n} X_k$: $d_\infty(x,y) = \max_{1 \le k \le n} d_k(x_k,y_k)$: d_∞ is equivalent to $d(x,y) = \left[\sum_{k=1}^{n} (d_k(x_k,y_k))^2\right]^{1/2}$ (the product metric) $d_\infty(x,y) \le d(x,y) \le d(x,y)\sqrt{n}$.

③ Pseudo metrics :

<u>Pseudometric on X</u>: X a nonempty set, $\rho : X \times X \longrightarrow [0, \infty)$ satisfying : • $\rho(x,y) \ge 0$, $\rho(x,y) = \rho(y,x)$, $\rho(x,y) \le \rho(x,z) + \rho(z,y)$
(X, ρ) is a pseudometric space. • $\boxed{x = y \Rightarrow \rho(x,y) = 0}$ So $x \neq y \Rightarrow \rho(x,y) = 0$ is possible!

<u>Prop</u>: (X, ρ) a pseudometric space $\Rightarrow (X/\sim, d)$ is a metric space where : $x \sim y \Leftrightarrow \rho(x,y) = 0$ $\forall x,y \in X$
$d([x],[y]) = \rho(x,y)$, for $[y]$ a class of equivalence in $X/\sim$ containing y.

(∵) • relat° of equiv : $x \sim x$; $x \sim y \Rightarrow y \sim x$; $x \sim y$ and $y \sim z \Rightarrow x \sim z$: $\rho(x,y) = 0$, $\rho(y,z) = 0 \Rightarrow 0 = \rho(x,y) + \rho(y,z) \ge \rho(x,z) \ge 0 \Rightarrow \rho(x,z) = 0$.
• <u>d is well def</u> : $\begin{cases} [x] = [x'] \\ [y] = [y'] \end{cases} \Rightarrow d([x],[y]) = d([x'],[y'])$: $\begin{cases} \rho(x,y) \le \rho(x,x') + \rho(x',y') + \rho(y',y) = 0 + \rho(x',y') + 0 \\ \rho(x',y') \le \rho(x',x) + \rho(x,y) + \rho(y,y') = 0 + \rho(x,y) + 0 \end{cases}$ (pseudometric) $\Rightarrow \rho(x,y) = \rho(x',y')$
$\Rightarrow d$ does not depend on the choice of representative
• <u>d is a metric</u> : $d([x],[y]) \ge 0$ as $\rho(x,y) \ge 0$, $d([x],[y]) = 0 \Leftrightarrow \rho(x,y) = 0 \Leftrightarrow x \sim y \Leftrightarrow [x] = [y]$, $d([x],[y]) = d([y],[x])$.
$d([x],[y]) = \rho(x,y) \le \rho(x,z) + \rho(z,y) = d([x],[z]) + d([z],[y])$ $\forall [x],[y],[z] \in X/\sim$

④ Countable products of metric spaces.

<u>Def</u>: A countable set of metric space : (X_k, d_k), $1 \le k \le n$: $\boxed{\begin{cases} X = \prod_{k=1}^{n} X_k = \{X = (x_k)_{k=1}^{\infty} : x_k \in X_k\} \\ d(x,y) = \sum_{k=1}^{n} 2^{-k} \underbrace{\frac{d_k(x_k,y_k)}{1 + d_k(x_k,d_k)}}_{<1} \end{cases}}$ $\rightarrow$ collect° of all sequences ST k^{th} element $\in X_k$
then (X,d) is a metric space when n is <u>finite</u>
(2^{-k}: decrease fast) $\Rightarrow d(x,y) < \infty$ $\forall x,y \in X$

(∵) $d(x,y) \ge 0$; $d(x,y) = d(y,x)$; $d(x,y) = 0 \Leftrightarrow \frac{d_k(x_k,y_k)}{1 + d_k(x_k,y_k)} = 0$ by positivity $\Leftrightarrow d_k(x_k,y_k) = 0$ $\forall 1 \le k \le n \Leftrightarrow x_k = y_k$ $\forall 1 \le k \le n \Leftrightarrow x = y$.

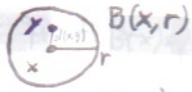
<u>Fact</u>: if $a,b,c \ge 0$: $a \le b + c \Rightarrow \frac{a}{1+a} \le \frac{b}{1+b} + \frac{c}{1+c}$
(∵) $\frac{a}{1+a} \le \frac{b}{1+b} + \frac{c}{1+c} \Leftrightarrow a \le b + c + \underbrace{(2bc + abc)}_{\ge 0}$, so $a \le b + c \Rightarrow \frac{a}{1+a} \le \frac{b}{1+b} + \frac{c}{1+c}$

$\rightarrow$ <u>Triangle ineq</u>: if $x,y,z \in X$, $d_k(x_k,y_k) \le d_k(x_k,z_k) + d_k(z_k,y_k)$
$\Rightarrow 2^{-k} \frac{d(x_k,y_k)}{1 + d(x_k,y_k)} \le 2^{-k} \frac{d(x_k,z_k)}{1 + d(x_k,z_k)} + 2^{-k} \frac{d(z_k,y_k)}{1 + d(z_k,y_k)}$
$\Rightarrow d(x,y) \le d(x,z) + d(z,y)$

(§9.2)

Open sets, closed sets & Convergence :

I Open sets :



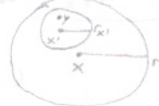
Let (X,d) be a metric space :

open ball = $x \in X$, $r>0$: $B(x,r) = \{y \in X : d(x,y) < r\}$ ⇝ open ball centered at x of radius r.

open set = $O \subset X$ is open if : $\forall x \in O,\ \exists r_x > 0$ ST $B(x, r_x) \subset O$ ⇝ At every pt $x \in O$, ∃ an open ball centered at x contained in O.

Neighborhood = $x \in X$: a neighborhood of x = open set containing x.

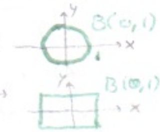
Prop : An open ball is an open set



Proof : Goal = $B(x,r)$ is open if : $\forall x' \in B(x,r)$, $\exists r_{x'} > 0$ ST $B(x', r_{x'}) \subset B(x,r)$

Take : $r_{x'} = r - d(x,x')$ ⇒ $\forall y \in B(x', r_{x'})$: $d(x,y) \le d(x,x') + \underbrace{d(x',y)}_{d(x',y) < r_{x'}} < d(x,x') + r_{x'} = r$: So if $y \in B(x', r_{x'}) \Rightarrow d(x,y) < r$ ⇒ $y \in B(x,r)$ ⇒ $B(x', r_{x'}) \subset B(x,r)$

Note : • The geometry of an open ball depends on the metric used : $X = \mathbb{R}^2$: → $\|x\| = \sqrt{x_1^2 + x_2^2}$ ⟶ ; → $\|x\| = \max(|x_1|, |x_2|)$ ⟶



• discrete metric : $\forall$ set X, $d(x,y) = \begin{cases} 0, & x=y \\ 1, & x \neq y \end{cases}$: $B(x,r) = \begin{cases} \{x\} & \text{if } r \le 1 \\ X & \text{if } r > 1 \end{cases}$

Prop : X itself is an open set, and $\emptyset$ is an open set.

Prop : (1) $\{O_i\}_{i \in I}$ (can be uncountable) a family of open sets in (X,d) ⟹ $\bigcup_{i \in I} O_i$ is an open set

(2) $\underbrace{O_1, \dots, O_n}_{\text{finite nb}}$ open sets ⇒ $O_1 \cap \dots \cap O_n$ is an open set.

Proof : (1) $x \in \bigcup_{i \in I} O_i$ ⇒ $x \in O_{i_0}$ for some $i_0 \in I$. But O_{i_0} open ⇒ $\exists r_x > 0$ ST $B(x, r_x) \subset O_{i_0} \subset \bigcup_{i \in I} O_i$ ⇒ $B(x, r_x) \subset \bigcup_{i \in I} O_i$

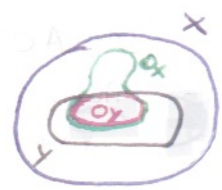
(2) $x \in O_1 \cap \dots \cap O_n$ ⇒ $x \in O_k$ $\forall 1 \le k \le n$. But O_k open ⇒ $\exists r_x^{(k)} > 0$ ST $B(x, r_x^{(k)}) \subset O_k$: Let $r_x = \min_{1 \le k \le n} \{r_x^{(k)}\}$ ⇒ $r_x > 0$ exists (since $\{r_x^{(k)}\}$ is finite) ⇒ $B(x, r_x) \subset B(x, r_x^{(k)})$ $\forall 1 \le k \le n$ as $r_x \le r_x^{(k)}$ ⇒ $B(x, r_x) \subset O_1 \cap \dots \cap O_n$.

⚠ (2) fails for countably many sets : $X = \mathbb{R}$, $d(x,y) = |x-y|$, $B(0, \frac{1}{k}) = (-\frac{1}{k}, \frac{1}{k})$
But $\bigcap_{k=1}^{\infty} B(0, \frac{1}{k}) = \{0\}$ is NOT an open set ! ⇝ $\nexists$ $B(0,r) = (-r, r) \subset \{0\}$ ST

Prop : O is an open set in (X,d) ⟺ $O = \bigcup_{i \in I} B(x_i, r_i)$

proof : (⇐) By previous prop : Union of open sets are open.

(⇒) $\forall x \in O$ open : $\exists r_x > 0$ ST $B(x, r_x) \subset O$ ⇒ $O \subset \bigcup_{x \in O} B(x, r_x) \subset O$ ⇒ $\bigcup_{x \in O} B(x, r_x) = O$



Prop : (Y,d) a metric subspace of (X,d) : $O_Y \subset Y$ is open ⟺ $O_Y = O_X \cap Y$ for some open $O_X \subset X$

Proof : $B_X(x,r) = \{y \in X : d(x,y) < r\}$ = open ball in X.

$B_Y(x,r) = \{y \in Y : d(x,y) < r\} = \{y \in X : d(x,y) < r\} \cap Y = B_X(x,r) \cap Y$ $\forall x \in Y$

So $B_Y(x,r) = B_X(x,r) \cap Y$ $\forall x \in Y$.

(⇒) Let $O_Y \subset Y$ an open set : by previous prop, $O_Y = \bigcup_{i \in I} B_Y(x_i, r_i) = \bigcup_{i \in I} (B_X(x_i, r_i) \cap Y) = \left(\bigcup_{i \in I} B_X(x_i, r_i)\right) \cap Y := O_X \cap Y$

So $O_X := \bigcup_{i \in I} B_X(x_i, r_i)$ is open in X by previous prop.

(⇐) Let $O_X \subset X$ an open set ; by previous prop, $O_X = \bigcup_{i \in I} B_X(x_i, r_i)$ ⇒ $O_X \cap Y = \bigcup_{i \in I} (B_X(x_i, r_i) \cap Y)$

• if $x_i \in Y$: $B_X(x_i, r_i) \cap Y = B_Y(x_i, r_i)$ is open in Y

(as) $B_X(x_i, r_i) \cap Y = \{x \in X : d(x, x_i) < r_i\} \cap \{x \in Y\} = \{x \in Y : d(x, x_i) < r_i\} = B_Y(x_i, r_i)$

• if $x_i \notin Y$: either $B_X(x_i, r_i) \cap Y = \emptyset$ → open in Y

OR $B_X(x_i, r_i) \cap Y = \{x \in Y : d(x, x_i) < r_i\} = B_Y(x_i, r_i')$ → open in Y.

⇒ $B_X(x_i, r_i) \cap Y$ open in Y ⇒ $O_Y = O_X \cap Y$ is open in Y

II Closed Set:

(X,d) a metric space:

<u>Closure point</u>: $A \subset X$: a point $x \in X$ is a closure pt of A $\iff$ $\forall$ neighborhood O of x, $O \cap A \neq \phi$.

$\rightsquigarrow$ every neighborhood of x contains a pt of A

<u>Closure of A</u>: set of all closure pts of A, denoted by: $\overline{A}$ or $cl(A)$.

Prop: $A \subset cl(A)$ $\quad$ $(\because)$ $x \in A, x \in O \Rightarrow x \in O \cap A \Rightarrow O \cap A \neq \phi \Rightarrow x \in cl(A)$

<u>closed set</u>: $A = cl(A) \Rightarrow A$ is closed

Prop: $cl(A)$ is the <u>smallest</u> <u>closed</u> set containing A (i.e.: $\forall$ set $F \supset A$, $F \supset cl(A)$)

<u>Proof</u>: • <u>$cl(A)$ is closed</u>: i.e. $cl(cl(A)) = cl(A)$

since $cl(A) \subset cl(cl(A))$ so we need to show that $cl(A) \supset cl(cl(A))$

$x \in cl(cl(A))$, O a neighborhood of x (open set containing x) $\Rightarrow O \cap cl(A) \neq 0 \Rightarrow \exists x' \in O \cap cl(A)$, so O is a neighborhood of x'

Since $x' \in cl(A)$ and O is a neighborhood of x' too $\Rightarrow O \cap A \neq \phi$

So $x \in cl(cl(A)) \Rightarrow \forall$ neighborhood O of x: $O \cap A \neq 0 \Rightarrow x \in cl(A)$ $\quad \Rightarrow cl(A) \supset cl(cl(A))$.

• <u>$cl(A)$ is the smallest closed set containing A</u>:

Let F be any closed set containing A: $x \in cl(A)$, O a neighborhood of $x \Rightarrow \phi = O \cap cl(cl(A)) = O \cap A$.

So $O \cap A \neq \phi$, and $O \cap F \neq \phi$ because $A \subset F$.

So $x \in cl(A) \Rightarrow \forall O$ neighboring x, $O \cap F \neq \phi \Rightarrow x \in cl(F) \Rightarrow x \in F$ as $\underset{F \text{ is closed}}{F = cl(F)} \Rightarrow cl(A) \subset F$.

Prop: $A \subset B \Rightarrow cl(A) \subset cl(B)$

<u>Proof</u>: $x \in cl(A) \Rightarrow \forall O$ neighboring x: $\phi \neq O \cap cl(cl(A)) = O \cap A$ $\Rightarrow \forall O$ neighboring x. $O \cap B \neq \phi$ $\quad (\because A \subset B)$

So $x \in cl(A) \Rightarrow x \in cl(B)$ $\quad \Rightarrow cl(A) \subset cl(B)$ $\quad \Rightarrow x \in cl(B)$

Prop: $A \subset X$ is closed $\iff A^c = X - A$ is open in X

<u>Proof</u>: $(\Rightarrow)$ $A \subset X$ is closed: So $A = cl(A)$ $\rightsquigarrow$ <u>Goal</u>: A^c is open

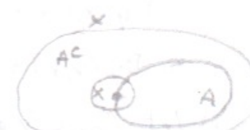
$\forall x \in A^c \Rightarrow x \notin A = cl(A) \Rightarrow x \notin cl(A) \Rightarrow x$ is <u>not</u> a closure pt of $A \Rightarrow \exists \underset{\text{open set containing } x}{O}$ neighboring x ST $O \cap A = \phi \Rightarrow O \subset A^c$.

So $\exists$ an open ball $B(x, r_x)$, $r_x > 0$ ST $B(x, r_x) \subset A^c$ $\Rightarrow A^c$ is open

$(\Leftarrow)$ A^c is open in X: let $x \in cl(A)$ $\rightsquigarrow$ Goal: $x \in A$

So $\forall$ neighborhood O of x: $O \cap A \neq \phi$. But in particular, $B(x,r)$ neighbors x $\forall r > 0$

$\Rightarrow \forall r > 0$, $B(x,r) \cap A \neq \phi$



A^c is closed: if $x \in A^c \Rightarrow \exists r > 0$ ST $B(x,r) \subset A^c \Rightarrow B(x,r) \cap A = \phi$. $\Rightarrow\Leftarrow$ So $x \notin A^c \Rightarrow x \in A$.

Hence $x \in cl(A) \Rightarrow x \in A$, so $cl(A) \subset A$, but $cl(A) \supset A \Rightarrow cl(A) = A \Rightarrow A$ is closed

Prop: • X is a closed set (and open too!) $\quad (\because)$ $X = X - \phi = \phi^c$

• ϕ is a closed set (and open too!) $\quad (\because)$ $\phi = X - X = X^c$

Prop: (1) $\{A_i\}_{i \in I}$ a $\overset{\text{not necess countable}}{\text{family}}$ of closed sets in $(X,d) \Rightarrow \bigcap_{i \in I} A_i$ is closed

(2) $A_1, \dots, A_n$ closed $\Rightarrow A_1 \cup \dots \cup A_n$ is closed.

<u>Proof</u>: (1) $\{A_i\}_{i \in I}$ closed $\Rightarrow \{A_i^c\}_{i \in I}$ open $\Rightarrow \bigcup_{i \in I} A_i^c$ open $\Rightarrow \left(\bigcup_{i \in I} A_i^c\right)^c$ is closed $\Rightarrow \bigcap_{i \in I} (A_i^c)^c = \bigcap_{i \in I} A_i$ is closed.

(2) $A_1, \dots, A_n$ closed $\Rightarrow A_1^c, \dots, A_n^c$ open $\Rightarrow A_1^c \cap \dots \cap A_n^c$ open $\Rightarrow (A_1^c \cap \dots \cap A_n^c)^c$ closed $\Rightarrow A_1 \cap \dots A_n$ closed.

Prop: $A, B \subset X$: $cl(A \cup B) = cl(A) \cup cl(B)$

III Interior & boundary of a set :

(X,d) a metric space, $A \subseteq X$

<u>Note</u>: $Cl(A)$ = smallest closed set containing A : $Cl(A) = \bigcap\limits_{\substack{F,\, F \supseteq A \\ F \text{ is closed}}} F$, so $Cl(A)$ always exists ($x \in A \Rightarrow x \in Cl(A)$)

<u>Interior of A</u> = largest open set contained in A : $int(A) = \bigcup\limits_{\substack{O,\, O \supset A \\ O \text{ open}}} O$, so $int(A)$ always exist, but could be open.

<u>Boundary of A</u> = $\partial A = \{x \in X : \forall$ neighborhood O of x, $O \cap A \neq \emptyset, O \cap A^c \neq \emptyset\} \Rightarrow \partial A = Cl(A) \cap Cl(A^c)$

<u>Prop</u>: ∂A is closed (as an intersect° of 2 closed sets)

<u>Prop</u>: $\partial A \cap int(A) = \emptyset$

<u>Proof</u>: $x \in int(A) \Rightarrow \exists r_x > 0$ ST $B(x, r_x) \subset int(A) \subset A \Rightarrow B(x, r_x) \cap A^c = \emptyset$ as $A \cap A^c = \emptyset$, Hence $x \notin \partial A$.

<u>Prop</u>: $Cl(A) = int(A) \cup \partial A$

<u>Proof</u>: • $x \in int(A) \cup \partial A \Rightarrow x \in \{\bigcup\limits_{\substack{O \supset A \\ O \text{ open}}} O\} \cup \{y \in X : \forall$ neighborhood O of y, $O \cap A \neq \emptyset, O \cap A^c \neq \emptyset\}$

• $x \in Cl(A)$: $\forall$ neighborhood O of x, $O \cap A \neq \emptyset$.

<u>Exercise</u> : (X,d) a metric space, $A, B \subset X$:

• $\partial(A \cup B) \subset (\partial A) \cup (\partial B)$: $\partial(A \cup B) = Cl(A \cup B) \cap Cl((A \cup B)^c) = (Cl(A) \cup Cl(B)) \cap (Cl(A^c \cap B^c)) = (Cl(A) \cap Cl(A^c \cap B^c)) \cup (Cl(B) \cap Cl(A^c \cap B^c))$
$A^c \cap B^c \subset A^c, B^c$, so $Cl(A^c \cap B^c) \subset Cl(A^c), Cl(B^c)$ $\Rightarrow \subset (Cl(A) \cap Cl(A^c)) \cup (Cl(B) \cap Cl(B^c)) = (\partial A) \cup (\partial B)$.

• Find example of X, A, B ST $\partial(A \cup B) \neq (\partial A) \cup (\partial B)$:
$X = \mathbb{R}$, $d(x,y) = |x-y|$, $A = (0,1)$, $B = (\frac{1}{2}, 2)$: $\partial A = \{0, 1\}$, $\partial B = \{\frac{1}{2}, 2\}$; $\partial(A \cup B) = \partial((0,2)) = \{0, 2\}$ $\rightarrow (\partial A) \cup \partial(B) = \{0, \frac{1}{2}, 1, 2\} \neq \{0, 2\} = \partial(A \cup B)$

• Find example of a countable collect° $\{A_n\}_{n=1}^{\infty}$, X ST $\bigcup\limits_{n=1}^{\infty} \partial A_n$ is properly contained in $\partial(\bigcup\limits_{n=1}^{\infty} A_n)$:
$X = \mathbb{R}$, $d(x,y) = |x-y|$, $\{r_n\}_{n=1}^{\infty}$ an enumeration of $\mathbb{Q}$: $A_n = \{r_n\} \Rightarrow \partial A_n = \{r_n\} \Rightarrow \begin{cases} \bigcup\limits_{n=1}^{\infty} A_n = \mathbb{Q} \Rightarrow \partial(\bigcup\limits_{n=1}^{\infty} A_n) = \partial \mathbb{Q} = \mathbb{R}. \\ \bigcup\limits_{n=1}^{\infty} \partial A_n = \mathbb{Q} \end{cases}$ $\mathbb{Q} \subsetneq \mathbb{R}$ ✓

(*) <u>$\partial \mathbb{Q} = \mathbb{R}$</u> : Let $r \in \mathbb{R}$, $B(r, \epsilon) = (r - \epsilon, r + \epsilon)$
By density thm, $B(r,\epsilon) \cap \mathbb{Q} \neq \emptyset$ (rational) and $B(r,\epsilon) \cap \mathbb{Q}^c \neq \emptyset$ (irrationals) $\Rightarrow r \in \partial \mathbb{Q}$

<u>Note</u> : X a nonempty set, $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$ the discrete metric

Any set in (X,d) is <u>both</u> open <u>and</u> closed !

<u>Proof</u>: • $A \subset X$ is open in (X,d) : $B(x, \frac{1}{2}) = \{x \in X : d(x,y) < \frac{1}{2}\} = \{x\} \Rightarrow A = \bigcup\limits_{x \in A} \{x\}$ is open as a union of open sets.

• $A \subset X$ is closed in (X,d) : $B(x, \frac{1}{2}) = \{x\} \Rightarrow A^c = \bigcup\limits_{x \in A^c} \{x\}$ is open $\Rightarrow (A^c)^c = A$ is closed in (X,d).

IV Convergence in metric Spaces:

Convergence of Sequences: A sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X,d) converges to a point $x \in X$:

$$\lim_{n\to\infty} x_n = x \iff \lim_{n\to\infty} d(x,x_n)=0 \quad \text{, i.e.: } \forall \varepsilon>0, \exists N>0 \text{ ST } d(x,x_n)<\varepsilon \ \forall n \geq N.$$

$\to$ x is called the limit (point) of the sequence (x_n).

Bounded Seq: $(x_n)_{n=1}^{\infty}$ in (X,d) is bounded if $\exists y \in X, M>0$ ST $|d(y,x_n)|<M \ \forall n\geq 1$. $\Rightarrow$ So $(d(y,x_n))_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$.

Note: this def is independent of the choice of y $(\circledast)$ $d(z,x_n) \leq d(z,y)+d(y,x_n) \leq \underbrace{d(z,y)+M}_{\text{finite and } \geq 0}$

Prop: A convergent sequence is bounded: $\lim_{n\to\infty} d(x,x_n)=0 \Rightarrow d(x,x_n)$ is bounded.

Proof: Take $\varepsilon=1$: $\exists N>0$ ST $d(x,x_n)<1 \ \forall n\geq N$ $\rightsquigarrow$ let $M=\operatorname{Max}\{d(x,x_1),\dots,d(x,x_N),1\} \Rightarrow d(x,x_n)\leq M. \ \forall n\geq 1$

Prop: If the limit exists, it is unique: $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y \Rightarrow x=y$ $(\circledast)$ $0 \leq d(x,y) \leq \underbrace{d(x,x_n)}_{\to 0}+\underbrace{d(x_n,y)}_{\to 0}$
$\Rightarrow$ Squeeze thm $d(x,y)=0 \Rightarrow x=y$

Prop: Let $A \subset X$: $x \in cl(A) \iff \exists$ a seq $(x_n)_{n=1}^{\infty}$, $x_n \in A$ ST $\lim_{n\to\infty} x_n = x$

Proof: $(\Rightarrow)$ $x \in cl$: So $\forall n>0$, $B(x,\frac{1}{n})$ is a neighborhood of x $\Rightarrow B(x,\frac{1}{n})\cap A \neq \phi \ \forall n>0$: take $x_n \in B(x,\frac{1}{n})\cap A$.
So $(x_n)_{n=1}^{\infty}$ is a sequence in A, and $0\leq d(x,x_n)\leq \frac{1}{n}\to 0 \Rightarrow \lim_{n\to\infty} d(x,x_n)=0 \Rightarrow (x_n)\to x$.

$(\Leftarrow)$ $x \in X$, $(x_n)_{n=1}^{\infty} \subset A$ ST $\lim_{n\to\infty} x_n = x$:
Let O a neighborhood of x: Take $\varepsilon>0$ ST $B(x,\varepsilon)\subset O$. So $x_n\to x \Rightarrow \exists N>0$ ST $x_n \in B(x,\varepsilon) \ \forall n\geq N \Rightarrow x_n \in O \ \forall n \geq N$.
So $x_n \in A$ and $x_n \in O \Rightarrow x_n \in O\cap A \Rightarrow O\cap A \neq \phi$ $\forall$ neighborhood O of $x \Rightarrow x \in cl(A)$.
$\forall n \geq N$

Prop: d_1, d_2 equivalent metrics on $X \Rightarrow (x_n)\to x$ in (X,d_1) and $(x_n)\to x$ in (X,d_2) $(\circledast)$ $\exists c_1,c_2>0$ ST: $c_1 d_2(x,y) \leq d_1(x,y) \leq c_2 d_2(x,y)$ $\forall x,y \in X$

Prop: d_1, d_2 equivalent metrics on $X \Rightarrow (X_1,d_1)$ and (X_2,d_2) have the same open / closed sets
i.e.: A open in $(X,d_1) \iff A$ open in (X,d_2) ; A closed in $(X,d_1) \iff A$ closed in (X,d_2) $\rightsquigarrow A \cdot cl(A)_{d_1} = cl(A)_{d_2}$

Proof: $A \subset X$: $x \in cl(A)_{d_1} \iff \exists (x_n) \in A$ ST $\lim_{n\to\infty} d_1(x_n,x)=0 \iff \exists (x_n) \in A$ ST $\lim_{n\to\infty} d_2(x_n,x)=0 \iff x \in cl(A)_{d_2}$

Convergence in product space: (X_k,d_k) a seq of metric spaces $(k=1,2,\dots)$

$$\begin{cases} X = \prod_{k=1}^{\infty} X_k = \{x=(x_k)_{k=1}^{\infty} : x_k \in X_k\} \\ d(x,y) = \sum_{k=1}^{\infty} 2^{-k} \dfrac{d_k(x_k,y_k)}{1+d_k(x_k,y_k)} \end{cases} : (x^{(n)})_{n=1}^{\infty} = \left((x_k^{(n)})_{k=1}^{\infty}\right)_{n=1}^{\infty} \text{ a sequence in } X$$

Prop: $x^{(n)}=(x_k^{(n)})_{k=1}^{\infty}$, $x=(x_k)_{k=1}^{\infty}$: $\lim_{n\to\infty} d(x^{(n)},x)=0 \iff \lim_{n\to\infty} d_k(x_k^{(n)},x_k)=0 \ \forall k\geq 1$
$\hookrightarrow x^{(n)} \to x$ as $n\to\infty$ $\hookrightarrow$ each component of the seq $x^{(n)}$ converges to the corresponding component of the seq x.

Proof: $(\Rightarrow)$ $\lim_{n\to\infty} d(x^{(n)},x)=0$:

$0 \leq 2^{-k}\dfrac{d_k(x_k^{(n)},x_k)}{1+d_k(x_k^{(n)},x_k)} \leq \underbrace{d(x^{(n)},x)}_{\to 0} \ \forall k\geq 1 \overset{\text{Squeeze thm}}{\Rightarrow} \lim_{n\to\infty} \dfrac{d_k(x_k^{(n)},x_k)}{1+d_k(x_k^{(n)},x_k)}=0 \Rightarrow 1=\lim_{n\to\infty}(1-(\ast\ast)) = \lim_{n\to\infty}\left(\dfrac{1}{1+d_k(x_k^{(n)},x_k)}\right)$
$\Rightarrow \lim_{n\to\infty} d_k(x_k^{(n)},x_k)=0 \ \forall k\geq 1$.

$(\Leftarrow)$ $\lim_{n\to\infty} d_k(x_k^{(n)},x_k)=0 \ \forall k\geq 1$:

$\forall \varepsilon>0$: • choose k_0 ST $\sum_{k=k_0}^{\infty} 2^{-k} < \frac{\varepsilon}{2}$

• $\lim_{n\to\infty} \sum_{k=1}^{k_0-1} 2^{-k}\dfrac{d_k(x_k^{(n)},x_k)}{1+d_k(x_k^{(n)},x_k)}=0 \Rightarrow \exists N_0>0$ ST $\sum_{k=1}^{k_0-1}(\ast\ast)<\frac{\varepsilon}{2} \ \forall n\geq N_0$

$\forall n \geq N_0$: $d(x^{(n)},x)=\sum_{k=1}^{k_0-1} 2^{-k}\dfrac{d_k(x_k^{(n)},x_k)}{1+d_k(x_k^{(n)},x_k)} + \sum_{k=k_0}^{\infty} 2^{-k}(\ast\ast) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

So $\forall \varepsilon>0$: $\exists N_0>0$ ST $d(x^{(n)},x)<\varepsilon \ \forall n\geq N_0 \rightarrow x^{(n)}\to x$ in (X,d).

V ℓ^p and L^p spaces :

① ℓ^p space :

$X = \mathbb{R}^n$, $x = (x_1, x_2, \dots, x_n)$: $\begin{cases} \|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}, & p \geq 1 \\ \|x\|_\infty = \max_{1\leq k\leq n} |x_k| \end{cases}$ are norms on $\mathbb{R}^n$: $\ell^p(\mathbb{R}) = (\mathbb{R}^n, \|\cdot\|_p)$ for $1 \leq p \leq \infty$

Prop: the norms $\|\cdot\|_p$ are equivalent $(1 \leq p \leq \infty)$: $\|\cdot\|_\infty \leq \|\cdot\|_1 \leq n\|\cdot\|_1$ (∵ $\mathbb{R}^n$ is a finite dim Vspace)

Def : • $\ell^p(\mathbb{N})$ = Vspace of all seq of real nb $x = (x_n)_{n=1}^\infty$ ST $\sum_{n=1}^\infty |x_n|^p < \infty$ ⇝ $\|\cdot\|_p$ is a norm on $\ell^p(\mathbb{N})$, $1 \leq p < \infty$
↳ $\|x\|_p = \left(\sum_{n} |x_n|^p\right)^{1/p}$

• $\ell^\infty(\mathbb{N})$ = Vspace of all bounded sequences $x = (x_n)_{n=1}^\infty$ $(x_n \leq M \ \forall n \geq 1)$ ⇝ $\|\cdot\|_\infty$ is a norm on $\ell^\infty(\mathbb{N})$
↳ $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$

② L^p space :

$X = C([a,b])$: Vspace of all cont f° $f : [a,b] \to \mathbb{R}$: $\begin{cases} \|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{1/p}, & p > 0 \\ \|f\|_\infty = \max_{x\in[a,b]} |f(x)| \end{cases}$ are norms on $C([a,b])$

Prop : (Hölder's inequality)

If $\begin{cases} 1 < p < \infty \\ 1 < q < \infty \end{cases}$ and $\frac{1}{p} + \frac{1}{q} = 1$: $\boxed{\int_a^b |f(x)| \cdot |g(x)| \, dx \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} \cdot \left(\int_a^b |g(x)|^q dx\right)^{1/q}}$

i.e: $\boxed{\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q}$

(∵) $\sum_{k=1}^n |x_k| \cdot |y_k| \leq \left(\sum_{k=1}^n |x_k|^p\right)^{1/p} \left(\sum_{k=1}^n |y_k|^q\right)^{1/q}$ (Proof in HW 2)

Prop : (Minkovski's Inequality)

$\forall p \geq 1$: $\boxed{\left(\int_a^b (|f(x)| + |g(x)|)^p dx\right)^{1/p} \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} + \left(\int_a^b |g(x)|^p dx\right)^{1/p}}$

i.e. : $\boxed{\|(|f| + |g|)\|_p \leq \|f\|_p + \|g\|_p}$

(∵) $\left(\sum_{k=1}^n (|x_k| + |y_k|)^p\right)^{1/p} \leq \left(\sum_{k=1}^n |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^n |y_k|^p\right)^{1/p}$ (Proof in HW 2)

Corr : $L^p = (C[a,b], \|\cdot\|_p)$ is a normed space for $p \geq 1$ ⇝ Minkovski ⇒ Triangle ineq
→ $\|\cdot\|_p$ is induced by an IP ⟺ $p = 2$ (∵ ~~//low~~ holds ⟺ $p=2$) ⇝ $p = 2$: $\langle f, g\rangle = \int_a^b fg \, dx$

Corr : $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p = \max_{x\in[a,b]} |f(x)|$

Proof : Let $M = \max_{x\in[a,b]} |f(x)|$: Goal : $\limsup_{p\to\infty} \|f\|_p \leq M \leq \liminf_{p\to\infty} \|f_p\|$

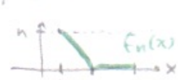
• $\limsup \|f\|_p \leq M$:
$\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{1/p} \leq \left(\int_a^b M^p dx\right)^{1/p} = M(b-a)^{1/p} \Rightarrow \limsup_{p\to\infty} \|f\|_p \leq M \cdot \limsup_{p\to\infty} (b-a)^{1/p} = M \cdot \limsup_{p\to\infty} e^{\frac{1}{p}\ln(b-a)} = M$.

• $\liminf \|f\|_p \geq M$:
· $\exists x_0 \in [a,b]$ ST $M = |f(x_0)|$: let $0 < \varepsilon < M$: f cont at x_0 ⇒ $\exists$ a closed interval $\Delta \subset [a,b]$ ST $x \in \Delta$ and $|f(x)| \geq M - \varepsilon \ \forall x \in \Delta$
(∵) $\exists \delta_0 > 0$ ST : "$\Delta = [x_0 - \delta, x_0 + \delta] \Rightarrow |f(x) - f(x_0)| < \varepsilon$" (Note : if $x_0 = a$: $\Delta = [x_0, x_0 + \delta]$; $x_0 = b$: $\Delta = [x_0 - \delta, x_0]$)
So $\varepsilon > |f(x) - f(x_0)| \geq |f(x_0)| - |f(x)| = M - |f(x)|$
$\Rightarrow |f(x)| \geq M - |f(x) - f(x_0)| > M - \varepsilon \quad \forall x \in \Delta$

· $\left(\int_a^b |f(x)|^p dx\right)^{1/p} \geq \left(\int_\Delta |f(x)|^p dx\right)^{1/p} \geq \left(\int_\Delta (M-\varepsilon)^p dx\right)^{1/p} = (M - \varepsilon)|\Delta|^{1/p}$ ($|\Delta|$ = length of Δ)

So $\liminf_{p\to\infty} \|f\|_p \geq (M - \varepsilon) \liminf_{p\to\infty} |\Delta|^{1/p} = M - \varepsilon \quad \forall \varepsilon > 0$ ⇒ take $\varepsilon \to 0$: $\liminf_{p\to\infty} \|f\|_p \geq M$

⚠ $\|\cdot\|_p$ and $\|\cdot\|_\infty$ are <u>NOT</u> equivalent : $\nexists D$ ST $\|f\|_p \geq D\|f\|_\infty \ \forall f \in C([a,b])$



$a_n = a + \frac{b-a}{n^{2p}}$

$\int_a^b |f_n(x)|^p dx \leq \int_a^{a_n} n^p dx = n^p(a_n - a) = \frac{b-a}{n^p}$

So $\left.\begin{array}{l} \|f_n\|_p \to 0 \\ \|f_n\|_\infty = n \to \infty \end{array}\right\}$ as $n \to \infty$

$\Rightarrow \|f\|_p \ngeq D\|f\|_\infty$ cannot be true $\forall f$!

Prop : $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent ⟺ $p = q$ ($p \neq q \Rightarrow \|\cdot\|_p, \|q\|$ not equiv!)

Proof : $p \neq q$: WLOG $p > q$:

Continuous mappings between Metric Spaces :

① Continuity : three definitions

(X, d_X), (Y, d_Y) metric spaces : $f : X \to Y$ a function

Continuous functions :

$x_n \to x \Rightarrow f(x_n) \to f(x) \quad \forall (x_n)_{n \geq 1}^{\infty}$

Continuity at x_0 : f is cont at $x_0 \in X$ if : $\boxed{\forall \text{ sequence } (x_n)_{n \geq 1}^{\infty} \text{ in } X, \ \lim_{n\to\infty} d_X(x_n, x) = 0 \Rightarrow \lim_{n\to\infty} d_Y(f(x_n), f(x)) = 0}$

Continuity : • f cont on $A \subset X$ if f cont at every pt $x_0 \in A$.
• f cont on X if f cont at every pt $x_0 \in X$.

$\to$ Sequential definition

Def : $f(A) = \{f(x) : x \in A\}$; $f^{-1}(B) = \{x \in X : f(x) \in B\}$ where $B \subset Y$

Prop : (1) f is continuous on $X \Rightarrow \forall x \in X : \forall \varepsilon > 0, \exists \delta > 0$ ST $f(B_X(x, \delta)) \subset B_Y(f(x), \varepsilon)$

$\Leftrightarrow$ (2) $\forall \varepsilon > 0, \exists \delta > 0$ ST $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon \quad \forall y \in X$.

$\Leftrightarrow$ (3) $\forall O \subset Y$ open, $f^{-1}(O) = \{x : f(x) \in O\}$ is an open set in X. $\leadsto$ Topological definition

Proof : • (1) $\Leftrightarrow$ (2) :

• (1) $\Rightarrow$ (2) : Suppose (2) is false $\Rightarrow \exists \varepsilon > 0$ ST $\forall \delta > 0, \exists y \in X$ ST $d_X(x, y) < \delta$ but $d_Y(f(x), f(y)) \geq \varepsilon$.
fix such an $\varepsilon_0 > 0$: take $d(x, y_n) = \frac{1}{n}$ and let $x_n \in B_X(x, \frac{1}{n})$ be such that $d_Y(f(x), f(x_n)) \geq \varepsilon_0$
then $d_X(x, x_n) < \frac{1}{n}$ and $d_Y(f(x), f(x_n)) \geq \varepsilon_0$
So $\lim_{n\to\infty} d_X(x, x_n) = 0$ and $\lim_{n\to\infty} d_Y(f(x_n), f(x)) \geq \varepsilon_0 \Rightarrow$ (1) fails , so (1) works $\Rightarrow$ (2) works.

• (2) $\Rightarrow$ (1) :
Let (x_n) be a seq ST $\lim_{n\to\infty} d_X(x_n, x) = 0$:
then $\forall \varepsilon > 0$, (2) $\Rightarrow \exists \delta > 0$ ST $(d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon)$
$\lim_{n\to\infty} d_X(x_n, x) = 0 \Rightarrow \exists N > 0$ ST $d_X(x, x_n) < \delta \ \forall n \geq N \Rightarrow d_Y(f(x), f(x_n)) < \varepsilon \ \forall n \geq N \Rightarrow \lim_{n\to\infty} d_Y(f(x), f(x_n)) = 0$

• (1) $\Leftrightarrow$ (3) :

• (1) $\Rightarrow$ (3) : O an open set in Y
$\to f^{-1}(O) = \emptyset \Rightarrow f^{-1}(O)$ is open in X ✓
$\to f^{-1}(O) \neq \emptyset \Rightarrow \exists x \in f^{-1}(O)$: So $f(x) \in O$, So $\exists \varepsilon > 0$ ST $B_Y(f(x), \varepsilon) \subset O$ as O is open.
f cont $\Rightarrow \exists \delta > 0$ ST $f(B_X(x, \delta)) \subset B_Y(f(x), \varepsilon) \subset O \Rightarrow \forall x \in f^{-1}(O), \exists \delta > 0$ ST $B_X(x, \delta) \subset f^{-1}(O)$
$\Rightarrow f^{-1}(O)$ is open.

• (3) $\Rightarrow$ (1) : Given $x \in X$, $\varepsilon > 0$:
$B_Y(f(x), \varepsilon)$ is open in Y ; So ③ $\Rightarrow f^{-1}(B_Y(f(x), \varepsilon))$ is open in X, with $x \in f^{-1}(B_Y(f(x), \varepsilon))$
So $\exists \delta > 0$ ST $B_X(x, \delta) \subset f^{-1}(B_Y(f(x), \varepsilon)) \Rightarrow \forall x \in X : \forall \varepsilon > 0, \exists \delta > 0$ ST $f(B_X(x, \delta)) \subset B_Y(f(x), \varepsilon)$

Prop : (X, d_X), (Y, d_Y), (Z, d_Z) metric spaces :

$\left.\begin{array}{l} f : X \to Y \\ g : Y \to Z \end{array}\right\}$ are continuous $\Rightarrow g \circ f : X \to Z$ is continuous on X.

Proof : Let $O \subset Z$ be open :
g cont $\Rightarrow g^{-1}(O)$ open in Y
f cont $\Rightarrow f^{-1}(g^{-1}(O))$ open in X
$\Bigg\}$ So $(g \circ f)^{-1}(O) = f^{-1}(g^{-1}(O))$ is open in $X \quad \forall O \subset Z$ open
$\Rightarrow g \circ f$ cont on X.

(1) Uniform continuity & Lipschitz functions:

(X,d_X), (Y,d_Y) metric spaces : $f: X \to Y$ a function

<u>Continuity</u> = $\forall x \in X : \forall \varepsilon > 0, \exists \delta(x,\varepsilon) > 0$ ST : $d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon \quad \forall y \in X$.

<u>uniform continuity</u> = $\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$ ST: $d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon \quad \forall x,y \in X$.

<u>Lipschitz Continuity</u> = $\exists c > 0$ ST : $d_Y(f(x), f(y)) \leq c\, d_X(x,y) \quad \forall x,y \in X$ $\rightsquigarrow$ Stronger than unif. cont.

<u>Prop</u>:
- f is Lipschitz $\Rightarrow$ f is uniformly continuous

  (∘∘) $\forall \varepsilon > 0$, take $\delta = \frac{\varepsilon}{c}$: $d_X(x,y) < \delta = \frac{\varepsilon}{c} \Rightarrow d_Y(f(x), f(y)) \leq c\, d_X(x,y) < c \cdot \frac{\varepsilon}{c} = \varepsilon \quad \forall x,y \in X$
- $f: [a,b] \to \mathbb{R}$, f diff, $|f'(x)|$ bounded $\Rightarrow$ f is Lipschitz

  (∘∘) $|f(x) - f(y)| \underset{\uparrow}{=} |f'(c)| \cdot |x-y| \leq M \cdot |x-y| \quad \forall x,y \in \mathbb{R}$

  Mean value thm. $\quad M = \max_{x \in [a,b]} |f'(c)|$

<u>Ex</u>:
- $X = (0,1]$: $f(x) = \frac{1}{x}$ $\to$ cont but not unif cont.
- $X \neq \phi$: $d_X(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases} \quad \forall x,y \in X$. Let (Y, d_Y) be any metric space : $f: X \to Y$ is always **unif cont** $\forall f$.

  (∘∘) $\delta = \frac{1}{2}$: $\forall \varepsilon > 0$, $d_X(x,y) < \frac{1}{2} \Rightarrow x = y \Rightarrow d_Y(f(x), f(y)) = 0 < \varepsilon$
- $X = \mathbb{R}$: $d_X(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$ on $\mathbb{R}$; $Y = \mathbb{R}$: $d_Y(x,y) = |x-y|$; $f: (\mathbb{R}, d_X) \to (\mathbb{R}, d_Y)$, $x \longmapsto x$ is unif cont but <u>not</u> Lipschitz

  (∘∘) suppose f is Lipschitz : $|f(x) - f(y)| \leq c\, d(x,y) \leq c \quad \forall x,y \in \mathbb{R}$, so $|x-y| \leq c \quad \forall x,y \in \mathbb{R}$ ⇒⇐
- $X = (C([a,b]), \|\cdot\|_p) \quad \forall 1 \leq p \leq \infty$: $F: X \to \mathbb{R}$, $F(f) = \int_a^b f(x)\,dx$ is Lipschitz. <u>Note</u>: F is a functional, and a linear operator.

  <u>$1 \leq p < \infty$</u>: $|F(f) - F(g)| = \left|\int_a^b (f-g)\,dx\right| \leq \int_a^b |f-g|\,dx \leq \left(\int_a^b |f-g|^p dx\right)^{1/p} \cdot \left(\int_a^b 1^q dx\right)^{1/q} \Rightarrow |F(f) - F(g)| \leq \|f-g\|_p \cdot (b-a)^{1/q}$

  <u>$p = \infty$</u>: $|F(f) - F(g)| = \left|\int_a^b (f-g)\,dx\right| \leq \int_a^b |f-g|\,dx \leq \int_a^b \max_{x\in[a,b]} |f-g|\,dx = \max_{x\in[a,b]} |f-g| \cdot \int_a^b 1\,dx = \max_{x\in[a,b]} |f-g| \cdot (b-a) \to |F(f) - F(g)| \leq \|f-g\|_\infty \cdot (b-a)$

  <u>F is Linear</u>: $F(\alpha f + \beta g) = \alpha F(f) + \beta F(g)$.
- $X = (C([a,b]), \|\cdot\|_p)$, $F: X \to \mathbb{R}$, $F(f) = \int_a^b |f(x)|^2 dx$ $\rightsquigarrow$ Not a linear operator!

  $|F(f) - F(g)| = \left|\int_a^b (f^2 - g^2)\,dx\right| = \left|\int_a^b (f-g)(f+g)\,dx\right| \leq \int_a^b |f-g| \cdot |f+g|\,dx$

  - F <u><u>is</u></u> Continuous $\forall 2 \leq p \leq \infty$: $\rightsquigarrow$ but neither unif cont nor Lipschitz!

    <u>$p=2$</u>: $\int_a^b |f-g| \cdot |f+g|\,dx \leq \left(\int_a^b |f-g|^2 dx\right)^{1/2} \left(\int_a^b |f+g|^2 dx\right)^{1/2} = \|f-g\|_2 \cdot \|f+g\|_2 \leq \|f-g\|_2 \cdot (\|f\|_2 + \|g_2\|) \longrightarrow |F(f) - F(g)| \leq \|f-g\|_2 \cdot (\|f\|_2 + \|g\|_2) \cdot \underbrace{1}_{:= c_2}$

    <u>$2 < p < \infty$</u>: $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{1}{q} = \frac{p-1}{p} \Rightarrow p = q(p-1)$

    $\int_a^b |f-g| \cdot |f+g|\,dx \leq \left(\int_a^b |f-g|^p dx\right)^{1/p} \left(\int_a^b |f+g|^q dx\right)^{1/q} = \|f-g\|_p \cdot \|f+g\|_q$

    Let q' : $qq' = p \Rightarrow q' = p-1$, $\frac{1}{p'} + \frac{1}{q'} = 1 \Rightarrow p' = \frac{p-1}{p-2}$

    $\left(\int_a^b |f+g|^q \cdot 1\,dx\right)^{1/q} \leq \left(\int_a^b |f+g|^{qq'} dx\right)^{\frac{1}{qq'}} \left(\int_a^b 1\,dx\right)^{\frac{1}{p'q}} = \|f+g\|_p \cdot (b-a)^{1/p'q}$

    $\longrightarrow |F(f) - F(g)| \leq \int_a^b |f-g| \cdot |f+g|\,dx \leq \|f-g\|_p \|f+g\|_p (b-a)^{\frac{1}{p'q}}$

    $\Rightarrow |F(f) - F(g)| \leq \|f-g\|_p (\|f\|_p + \|g\|_p) \underbrace{(b-a)^{\frac{1}{p'q}}}_{:= c_p}$ with $\frac{1}{p'q} = \frac{p-2}{p} = 1 - \frac{2}{p}$

    <u>$p = \infty$</u>: $\int_a^b |f-g| \cdot |f+g|\,dx \leq \int_a^b \max_{x\in[a,b]} |f-g| \cdot |f+g|\,dx = \|f-g\|_\infty \int_a^b |f+g|\,dx \leq \|f-g\|_\infty \int_a^b \max_{x\in[a,b]} |f+g|\,dx = \|f-g\|_\infty \cdot \|f+g\|_\infty \int_a^b 1\,dx \leq \|f-g\|_\infty \cdot (\|f\|_\infty + \|g\|_\infty) \cdot (b-a)$

    So $|F(f) - F(g)| \leq \|f-g\|_\infty \cdot (\|f\|_\infty + \|g\|_\infty) \cdot \underbrace{(b-a)^1}_{:= c_\infty}$

    <u>So</u>: $\boxed{|F(f) - F(g)| \leq \|f-g\|_p \cdot (\|f\|_p + \|g\|_p) \cdot c_p \quad \text{where } c_p = (b-a)^{1-\frac{2}{p}} \quad \forall 2 \leq p \leq \infty}$

    $\left(c_2 = (b-a)^{1-\frac{2}{2}} = (b-a)^{1-1} = (b-a)^0 = 1 \right.$, $\left. c_\infty = (b-a)^{1-\frac{2}{\infty}} = (b-a)^{1-0} = (b-a) \right)$

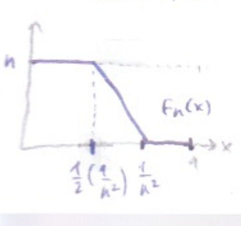
    <u>Continuity</u>: $(f_n)_{n=1}^\infty$ a seq in $C([a,b])$ converging to g in $\|\cdot\|_p$ norm : $\lim_{n\to\infty} \|f_n - g\|_p = 0 \Rightarrow \lim_{n\to\infty} |F(f_n) - F(g)| = 0$

    $|F(f_n) - F(g)| \leq \|f_n - g\|_p \cdot \underbrace{(\|f_n\|_p + \|g\|_p) \cdot c_p}_{\leq M \ \forall n \geq 1}$ since (f_n) converges $\Rightarrow \|f_n\|_p$ is bounded; g is fixed so $\|g\|_p$ is bounded; $(f_n - g)$ converges $\Rightarrow \|f_n - g\|$ is bounded ; c_p is bounded.

    $\to \lim_{n\to\infty} F(f_n) = F(g) \quad \forall f_n \to g$ in $(C([a,b]), \|\cdot\|_p) \Rightarrow F$ is cont $\forall 2 \leq p \leq \infty$.

  - F is <u><u>NOT</u></u> Continuous $\forall 1 \leq p < 2$:

    take $a=0, b=1$



$\int_0^1 |f_n(x)|^p dx \leq \int_0^{1/n^2} n^p\,dx = n^p \cdot \frac{1}{n^2} = \left(\frac{1}{n}\right)^{2-p} \to 0$ as $n \to \infty$ for $1 \leq p < 2$

$F(f_n) = \int_0^1 |f_n(x)|^2 dx \geq \int_0^{\frac{1}{2}\cdot\frac{1}{n^2}} n^2\,dx = n^2 \cdot \left(\frac{1}{2} \cdot \frac{1}{n^2}\right) = \frac{1}{2}$

$f_n \to 0$ in $\|\cdot\|_p \quad \forall 1 \leq p < 2$

<u>BUT</u> $\liminf_{n\to\infty} F(f_n) \geq \frac{1}{2} \neq F(0)$

$\rightsquigarrow$ F is <u>not</u> cont at 0 $\Rightarrow$ F is <u>NOT</u> Continuous

Complete Metric Spaces:

(I) Cauchy Sequences:

<u>Cauchy Seq</u> = (X, d) a metric space : a sequence $(x_n)_{n=1}^{\infty}$ in X is called Cauchy if :
$\forall \epsilon > 0, \exists N > 0$ ST $d(x_n, x_m) < \epsilon \quad \forall n, m \geq N$.

Prop: (1) A Converging Sequence is Cauchy : (x_n) converges $\Rightarrow (x_n)$ is Cauchy
(2) (x_n) is cauchy <u>and</u> has a converging subsequence $\Rightarrow (x_n)$ converges. $\rightsquigarrow$ if (x_n) is Cauchy <u>AND</u> <u>not</u> Convergent $\Rightarrow$ <u>No</u> Subseq can converge!

<u>Proof</u>: (1) Let $\lim_{n\to\infty} x_n = x \Rightarrow \underline{\forall \epsilon > 0, \exists N > 0 \text{ ST}}\ d(x_n, x) < \frac{\epsilon}{2} \quad \forall n \geq N$
$\Rightarrow \underline{\forall n, m \geq N}: \underline{d(x_n, x_m)} \leq d(x_n, x) + d(x, x_m) < \underline{\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon}$

(2) (x_n) Cauchy, $(x_{n_k})_{k=1}^{\infty}$ a convergent Subseq:
(x_n) cauchy $\Rightarrow \underline{\forall \epsilon > 0, \exists N > 0 \text{ ST}}\ d(x_n, x_m) < \frac{\epsilon}{2} \quad \forall n, m \geq N$
$\lim_{k\to\infty} x_{n_k} = x \Rightarrow \exists n_{k_0} > N$ ST $d(x_{n_{k_0}}, x) < \frac{\epsilon}{2}$
$\Bigg\}$ So for $\underline{n \geq N}$:
$\underline{d(x_n, x)} \leq d(x_n, x_{n_{k_0}}) + d(x_{n_{k_0}}, x) < \underline{\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon}$

⚠ (x_n) Cauchy $\not\Rightarrow$ (x_n) Converges

(II) Complete, Banach & Hilbert Spaces:

<u>Completeness</u> = A metric space (X, d) is complete if : Every Cauchy Seq in X converges to an element in X.

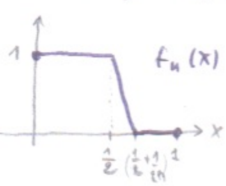
<u>Ex</u>: • $(\mathbb{R}, d)$, $d(x,y) = |x - y|$ is a <u>complete</u> metric space.
• $X \neq \phi$ a nonempty set, $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$ the discrete metric : (X, d) is complete
(∵) (x_n) a Cauchy Seq in (X,d) : $\epsilon = \frac{1}{2} \Rightarrow \exists N > 0$ ST $d(x_m, x_n) < \frac{1}{2} \quad \forall m, n \geq N \Rightarrow d(x_m, x_n) = 0 \quad \forall n, m \geq N \Rightarrow x_m = x_n \quad \forall n, m \geq N$
$\rightarrow$ So any Cauchy Sequence is constant for large index $\Rightarrow$ any Cauchy Seq converges in $(X,d) \Rightarrow$ COMPLETE.

<u>Banach Space</u> = $(X, \|\cdot\|)$ a normed space : if $d(x,y) = \|x - y\|$ is complete then $(X, \|\cdot\|)$ is a Banach Space.

<u>Hilbert Space</u> = $(X, \|\cdot\|)$ a Banach space : if $\|\cdot\|$ is induced by an IP ($\|x\| = \sqrt{\langle x, x\rangle}$), then $(X, \|\cdot\|)$ is a Hilbert Space

Prop: (a) $(C([a,b]), \|\cdot\|_p)$ is <u>NOT</u> complete for $1 \leq p < \infty$
(b) $(C([a,b]), \|\cdot\|_\infty)$ is a Banach Space.

<u>Proof</u>: (a) <u>Case $p = 1$</u>: let $a = 0, b = 1$ ($\rightarrow p > 1$: Exercise!)



• $-1 \leq f_n(x) - f_m(x) \leq 1 \quad \forall x \in [0,1]$ and : $\begin{cases} f_n(x) - f_m(x) = 0 \text{ if } 0 \leq x \leq 1/2 \\ f_n(x) - f_m(x) = 0 \text{ if } \frac{1}{2} + \max\{\frac{1}{2m}, \frac{1}{2n}\} \leq x \end{cases}$

So $\int_0^1 |f_n(x) - f_m(x)|\, dx = \int_{\frac{1}{2}}^{\frac{1}{2} + \max\{\frac{1}{2m}, \frac{1}{2n}\}} |f_n(x) - f_m(x)|\, dx \leq \int 1\, dx = \max\{\frac{1}{2m}, \frac{1}{2n}\}$

So $\forall \epsilon > 0$, take $N > \frac{1}{2\epsilon}$: then $\int_0^1 |f_n(x) - f_m(x)|\, dx \leq \frac{1}{2N} < \epsilon \Rightarrow \boxed{(f_n) \text{ is Cauchy in } (C([0,1]), \|\cdot\|_1)}$

• <u>Goal</u>: $f_n \to g \notin (C[0,1], \|\cdot\|_1)$ because g is <u>not</u> continuous!

If $f_n \to g$ in $(C[0,1], \|\cdot\|_1)$: $\lim_{n\to\infty} \int_0^1 |f_n - g|\, dx = 0 \Rightarrow 0 = \lim_{n\to\infty} \int_0^{1/2} |f_n - g|\, dx = \int_0^{1/2} |1 - g|\, dx \Rightarrow g \equiv 1 \quad \forall 0 \leq x \leq 1/2$

let $\frac{1}{2} < t < 1$: $\int_0^1 |f_n - g|\, dx \geq \int_t^1 |f_n - g|\, dx \geq 0 \Rightarrow 0 = \lim_{n\to\infty} \int_t^1 |f_n - g|\, dx = \lim_{n\to\infty} \int_t^1 |0 - g|\, dx = \int_t^1 g\, dx \Rightarrow g \equiv 0 \quad \forall x \in [t, 1]$ with $t > 1/2$.

($f_n \equiv 0$ on $[t,1]$ for n large enough)

So $g(x) = \begin{cases} 1, & 0 \leq x \leq 1/2 \\ 0, & \frac{1}{2} < x \leq 1 \end{cases} \Rightarrow \boxed{g \text{ is NOT continuous on } [0,1]}$

Proof: (b) $(f_n)_{n=1}^{\infty}$ a Cauchy seq in $(C([a,b]), \|\cdot\|_\infty) \Rightarrow \forall \epsilon>0, \exists N>0$ ST $\max_{a\le x\le b} |f_n(x)-f_m(x)| = \|f_n - f_m\|_\infty < \epsilon \ \forall n,m \ge N$

- Claim: $\exists f$ ST $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$

(*) (f_n) Cauchy $\Rightarrow \forall x\in[a,b]: \forall \epsilon>0, \exists N>0$ ST $|f_m(x)-f_n(x)| \le \|f_n-f_m\|_\infty < \epsilon \ \forall n,m\ge N \Rightarrow (f_n(x))$ is a Cauchy seq of real numbers

$(\mathbb{R},|\cdot|)$ is complete so $\exists f$ ST $\lim_{n\to\infty} f_n(x) = f(x) \ \forall x\in[a,b]$.

- Claim: $f\in C([a,b])$, i.e f is cont: $(\forall x_0\in[a,b])(\forall\epsilon>0)(\exists\delta>0)(\forall y\in[a,b])\big(|x_0-y|<\delta \Rightarrow |f(x_0)-f(y)|<\epsilon\big)$

(*) Let $x_0\in[a,b]$: Given $\epsilon>0$, $\exists N\ge 0$ ST $|f_m(x)-f_n(x)| \le \max_{x\in[a,b]} |f_m(x)-f_n(x)| < \frac{\epsilon}{3} \ \forall n,m\ge N$ as f_n is Cauchy.

Fix $m\ge N$, let $n\to\infty \Rightarrow |f_m(x)-f(x)| \le \frac{\epsilon}{3} \ \forall m\ge N, \forall x\in[a,b] \Rightarrow \boxed{|f_N(x)-f(x)| \le \frac{\epsilon}{3}, \forall x\in[a,b]}$ → also true for $x=x_0$

f_n is cont for each $n \Rightarrow f_N(x)$ is cont $\Rightarrow \exists\delta>0$ ST if $y\in[a,b]: |x_0-y|<\delta \Rightarrow \boxed{|f_N(x_0)-f_N(y)| < \frac{\epsilon}{3}}$

So if $|x_0-y|<\delta$: $|f(x_0)-f(y)| = |f(x_0)-f_N(x_0)+f_N(x_0)-f_N(y)+f_N(y)-f(y)|$
$\forall y\in[a,b]$

$\le |f(x_0)-f_N(x_0)| + |f_N(x_0)-f_N(y)| + |f_N(y)-f(x)| < \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon$.

- Claim: $f_n \to f$ in $(C([a,b]), \|\cdot\|_\infty)$

Given $\epsilon>0$: $\exists N>0$ ST $\forall m,n\ge N$: $\|f_m-f_n\|_\infty = \max_{a\le x\le b} |f_m(x)-f_n(x)| < \frac{\epsilon}{2} \Rightarrow |f_m(x)-f_n(x)| < \frac{\epsilon}{2} \ \forall x\in[a,b]$

take $m\to\infty$: $|f_n(x)-f(x)| \le \frac{\epsilon}{2} \ \forall x\in[a,b], \forall n\ge N \Rightarrow \max_{a\le x\le b}|f_n(x)-f(x)| \le \frac{\epsilon}{2} < \epsilon \Rightarrow \|f_n(x)-f(x)\|_\infty < \epsilon$

$\Rightarrow f_n \to f$ in $(C([a,b]), \|\cdot\|)$

III Complete metric Subspaces:

⤳ in general: a subspace of a complete metric space is NOT complete

↳ Ex: $(\mathbb{R},|\cdot|)$ complete but not $((0,1),|\cdot|) \subset (\mathbb{R},|\cdot|)$

Prop: (X,d) a complete metric space: $(Y,d)\subset(X,d)$ is complete $\iff$ Y is a closed subset of X.

Proof: ($\Leftarrow$) Y is closed: Let $(x_n)_{n=1}^{\infty}$ be a Cauchy seq in $Y\subset X \Rightarrow (x_n)_{n=1}^{\infty}$ is Cauchy in X : X complete $\Rightarrow \lim_{n\to\infty} x_n = x \in X$ $\exists$

Y closed: if $x_n\in Y$ then $x\in Cl(Y)=Y \Rightarrow \lim_{n\to\infty} x_n = x \in Y \Rightarrow Y$ is complete!

($\Rightarrow$) Y is complete: Let $x\in Cl(Y)$: then $\exists (x_n)_{n=1}^{\infty}$ in Y that converges to x.

(x_n) converges $\Rightarrow (x_n)$ is Cauchy $\Rightarrow x_n\to x\in Y$ because Y is complete $\Rightarrow Cl(Y)\subset Y \Rightarrow Y = Cl(Y) \Rightarrow Y$ is closed

diameter: (X,d) a metric space: $(Y,d)\subset(X,d)$ nonempty : $diam(Y) = \sup\{d(x,y) : x,y\in Y\}$

→ Y is bounded if $diam(Y)<\infty$

Prop: $diam(Y) = diam(Cl(Y))$

Proof:
- $Y\subset Cl(Y) \Rightarrow diam(Y) \subset diam(Cl(Y))$
- $x,y\in Cl(Y)$: $\exists$ 2 seq in Y: $(x_n)_{n=1}^{\infty}\to x$ and $(y_n)_{n=1}^{\infty}\to y \Rightarrow \lim_{n\to\infty} d(x_n,x) = 0 = \lim_{n\to\infty} d(y_n,y)$

So $\begin{cases} d(x,y) \le d(x,x_n)+d(x_n,y_n)+d(y_n,y) \\ d(x_n,y_n) \le d(x_n,x)+d(x,y)+d(y_n,y) \end{cases} \Rightarrow \begin{cases} d(x,y)-d(x_n,y_n) \le d(x,x_n)+d(y,y_n) \\ d(x_n,y_n)-d(x,y) \le d(x,x_n)+d(y,y_n) \end{cases} \Rightarrow |d(x,y)-d(x_n,y_n)| \le d(x,x_n)+d(y,y_n)$

$\Rightarrow \lim_{n\to\infty} d(x_n,y_n) = d(x,n)$

So $diam(Y) = \sup\{d(x,y): x,y\in Y\} \ge d(x_n,y_n) \ \forall n>0$

So $diam(Y) \ge d(x,y) \ \forall x,y\in Cl(Y)$ $\Rightarrow diam(Y) \ge \lim_{n\to\infty} d(x_n,y_n) = d(x,y)$

$\Rightarrow diam(Y)$ is an upperbound for $\{d(x,y): x,y\in Cl(Y)\}$

$\Rightarrow diam(Y) \ge \sup\{d(x,y): x,y\in Cl(Y)\} = diam(Cl(Y))$

IV Cantor intersection theorem:

Contracting Sequence: A descending sequence of nonempty subsets of a metric space (X,d): $Y_1 \supset Y_2 \supset \dots \supset Y_n \supset Y_{n+1} \supset \dots$ is called contracting if: $\lim_{n\to\infty} \operatorname{diam}(Y_n) = 0$

Cantor intersect° thm: the metric space (X,d) is Complete ① $\iff$ ② for any contracting seq. (F_n) of nonempty closed subsets of X, $\exists x \in X$ ST: $\bigcap_{n=1}^{\infty} F_n = \{x\}$

Proof: ($\Rightarrow$) Let (F_n) be as in ②: since $F_n \neq \phi$, choose $x_n \in F_n$

- claim: (x_n) is Cauchy

  $\varepsilon>0$ given: $\lim_{n\to\infty} \operatorname{diam}(F_n) = 0 \Rightarrow \exists N>0$ ST $\operatorname{diam}(F_N) < \varepsilon$.

  $\forall n,m \geq N$: $x_n \in F_n \subset F_N$, $x_m \in F_m \subset F_N \Rightarrow d(x_n,x_m) \leq \operatorname{diam}(F_N) < \varepsilon \Rightarrow (x_n)$ is Cauchy.

- claim: $\{x\} \subset \bigcap_{n=1}^{\infty} F_n$

  X is complete and (x_n) is Cauchy $\Rightarrow \lim_{n\to\infty} x_n = x \in X$ exists.

  $(x_n)_{n=1}^{\infty} \in F_n \subset F_1 \Rightarrow (x_n)_{n=1}^{\infty}$ is a seq in F_1, and F_1 closed $\Rightarrow x \in F_1$

  $(x_n)_{n=2}^{\infty} \in F_n \subset F_2 \Rightarrow (x_n)_{n=2}^{\infty}$ is a seq in F_2, and F_2 closed $\Rightarrow x \in F_2$

  $(x_n)_{n=k}^{\infty} \in F_n \subset F_k \Rightarrow (x_n)_{n=k}^{\infty}$ is a seq in F_k, and F_k closed $\Rightarrow x \in F_k \ \forall k \geq 1 \Rightarrow \{x\} \subset \bigcap_{k=1}^{\infty} F_k$

- claim: $\{x\} = \bigcap_{n=1}^{\infty} F_n$

  if $y \in \bigcap_{n=1}^{\infty} F_k$, then $x,y \in F_k \ \forall k \geq 1$, and $d(x,y) \leq \operatorname{diam}(F_k)$

  (F_k) contracting $\Rightarrow \lim_{k\to\infty} \operatorname{diam}(F_k) = 0 \Rightarrow d(x,y) = 0 \Rightarrow x = y \Rightarrow \{x\} = \bigcap_{n=1}^{\infty} F_n$

($\Leftarrow$) $(x_n)_{n=1}^{\infty}$ a Cauchy seq in X:

Let $F_k := Cl\{x_n : n \geq k\} \Rightarrow F_k$ is closed and $F_{k+1} \subset F_k$ as $\{x_n : n \geq k\} \supset \{x_n : n \geq k+1\}$

- claim: $\lim_{n\to\infty} \operatorname{diam}(F_n) = 0$

  $\varepsilon>0$ given: x_n Cauchy in $X \Rightarrow \exists N>0$ ST $\forall m,n \geq N$, $d(x_n,x_m) < \frac{\varepsilon}{2}$

  so $\operatorname{diam}(Cl\{x_n : n \geq N\}) = \operatorname{diam}(\{x_n : n \geq N\}) = \sup\{d(x_n,x_m) : n,m \geq N\} \leq \frac{\varepsilon}{2} < \varepsilon \Rightarrow \operatorname{diam} F_N < \varepsilon$

  (F_n) is descending $\Rightarrow F_n \subset F_N \ \forall n \geq N \Rightarrow \operatorname{diam}(F_n) \leq \operatorname{diam}(F_N) < \varepsilon \ \forall n \geq N \Rightarrow \lim_{n\to\infty} \operatorname{diam}(F_n) = 0$

  $\Rightarrow (F_n)$ is contracting

- claim: $(x_n) \to x \in X$, i.e. X is complete

  (F_n) is a contracting seq of nonempty closed subsets of $X \Rightarrow \exists x \in X$ ST $\bigcap_{n=1}^{\infty} F_n = \{x\}$

  So $\forall k \geq 1$: $x_k \in F_k$ and $x \in F_k \Rightarrow 0 \leq d(x_k,x) \leq \operatorname{diam}(F_k) \to 0$ as $k\to\infty \Rightarrow \lim_{k\to\infty} d(x_k,x) = 0$

  $\Rightarrow \lim_{k\to\infty} x_k = x \in X$

  So (x_n) converges in $X \Rightarrow X$ is complete.

V Completion of metric spaces :

Isometric metric spaces = The metric spaces (X, d_X) and (Y, d_Y) are isometric if $\exists$ a bijection $f: X \to Y$, ST $\forall x, y \in X$
$d_Y(f(x), f(y)) = d_X(x, y)$

Dense subset = A subset S of (X, d) is dense if $cl(S) = X$
i.e. $\forall x \in X$, $\exists$ a seq $(x_n) \in S$ ST $\lim_{n\to\infty} x_n = x$

Ex: $(\mathbb{Q}, |\cdot|)$ is not complete, but $\mathbb{Q}$ is dense in $\mathbb{R}$.

Prop: $X \neq \phi$, d the discrete metric on X: $S \subset (X, d)$ is dense in X $\Rightarrow S = X$

(pf) S dense in $X \Rightarrow \forall x \in X: \exists (x_n) \in S$ and $\exists N \in \mathbb{N}$ ST $d(x_n, x) < \frac{1}{2}$ $\forall n \geq N$ $\Rightarrow x_n = x$ $\forall n \geq N$. $\Rightarrow x \in S$.
$\Rightarrow$ if $x \in X$ then $x \in S$ $\Rightarrow X \subset S$, but $S \subset X$ by assumpt°, so $X = S$.

Completion of a metric space = A complete metric space (Y, d_Y) is a completion of (X, d_X) if:
$\exists$ a subset $\tilde{Y} \subset Y$ ST: (i) $\tilde{Y}$ is dense in Y
(ii) $\exists$ a bijecti° $f: X \to \tilde{Y}$ ST: $\forall x, y \in X$, $d_Y(f(x), f(y)) = d_X(x, y)$
(iii) Y is complete

$\to$ (X, d_X) is isometric to a dense metric subspace of (Y, d_Y)

Thm: (1) Any metric space has a completion
(2) A completion of a metric space is unique up to isometry: i.e. $(Y, d_Y), (Z, d_Z)$ completions of (X, d_X) $\Rightarrow (Y, d_Y) \cong (Z, d_Z)$
(3) The completion of a normed space is a Banach space.
(4) The completion of a normed space where the norm is induced by an IP is a Hilbert space

Prop: • A finite collect° of metric spaces $\{(X_k, d_k)\}_{k=1}^N$, $X = \prod_{k=1}^N X_k$, $d(x,y) = \sum_{k=1}^N 2^{-k} \cdot \frac{d(x_k, y_k)}{1 + d(x_k, y_k)}$:
(X, d) is complete $\Leftrightarrow$ (X_k, d_k) is complete $\forall 1 \leq k \leq n$

• A countable collect° of metric spaces $\{(X_k, d_k)\}_{k=1}^\infty$, $X = \prod_{k=1}^\infty X_k$, $d(x,y) = \sum_{k=1}^\infty 2^{-k} \cdot \frac{d(x_k, y_k)}{1 + d(x_k, y_k)}$:
(X, d) is complete $\Leftrightarrow$ (X_k, d_k) is complete $\forall k \geq 1$

Corr: $\mathbb{R}^n$ is complete

Proof: • Let $(x^{(n)})_{n=1}^\infty = (x_k^{(n)})_{k=1}^N$ for $n = 1, 2, \dots$ be a Cauchy seq in (X, d) : Goal = $(x^{(n)})_{n=1}^\infty$ converges in (X, d)

Note: $(X^{(n)})_{n=1}^\infty = (x_1^{(n)}, \dots, x_N^{(n)})_{n=1}^\infty$ converges in (X, d) $\Leftrightarrow$ $x_1^{(n)}, \dots, x_N^{(n)}$ converge in $(X_1, d_1), \dots, (X_N, d_N)$ resp

$\Rightarrow$ Goal: $(x_k^{(n)})_{n=1}^\infty$ converge in (X_k, d_k) for each $1 \leq k \leq n$:

Fix k $\longrightarrow$ claim: $(x_k^{(n)})_{n=1}^\infty$ is Cauchy in (X_k, d_k), i.e. $\exists N > 0$ ST $d_k(x_k^{(n)}, x_k^{(m)}) < \varepsilon$ $\forall n, m \geq N$.

$\varepsilon > 0$ given: let $\varepsilon' > 0$ ST $0 < \frac{2^k \varepsilon'}{1 - 2^k \varepsilon'} < \varepsilon$:

$(X^{(n)})_{n=1}^\infty$ is Cauchy $\Rightarrow \exists N > 0$ ST $\forall m, n \geq N$: $d(x^{(n)}, x^{(m)}) = \sum_{j=1}^N 2^{-j} \frac{d_j(x_j^{(n)}, x_j^{(m)})}{1 + d_j(x_j^{(n)}, x_j^{(m)})} < \varepsilon'$ $\forall n, m \geq N$

So in particular: $2^{-k} \frac{d_k(x_k^{(n)}, x_k^{(m)})}{1 + d_k(x_k^{(n)}, x_k^{(m)})} < \varepsilon'$ $\Rightarrow d_k(x_k^{(n)}, x_k^{(m)}) < \frac{2^k \varepsilon'}{1 - 2^k \varepsilon'} < \varepsilon$ $\forall n, m \geq N$

So $(x_k^{(n)})_{n=1}^\infty$ is Cauchy in (X_k, d_k) for each $1 \leq k \leq N$

$\Rightarrow$ Since (X_k, d_k) is complete for each $1 \leq k \leq N$ and $(x_k^{(n)})_{n=1}^\infty$ is Cauchy in (X_k, d_k) for each $1 \leq k \leq N$ $\Rightarrow$ $(x_k^{(n)})_{n=1}^\infty$ converge in (X_k, d_k) for each $1 \leq k \leq n$.

• Same proof: write $N = \infty$ instead

Thm: (Extension by uniform continuity)

Let (X, d_X), (Y, d_Y) metric spaces, (Y, d) is complete
Let $S \subset X$ be dense in X ($\hookrightarrow cl(S) = X$), and let $f : S \to Y$ be a unif cont f° ($\Leftrightarrow (\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in S)(d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon)$)
$\Rightarrow \exists!$ unif cont f° $\bar{f} : X \to Y$ ST $\bar{f}(x) = f(x)\ \forall x \in S$

Proof:

- **Uniqueness:** Suppose $\bar{f}$ and $\bar{g}$ satisfy the cond° of the thm:
  Let $x \in X$ and $(x_n)_{n=1}^{\infty}$ a sequence in S converging to x (as $cl(S) = X$) $\Rightarrow \bar{f}(x_n) = f(x_n) = \bar{g}(x_n)$
  $\bar{f}, \bar{g}$ cont $\Rightarrow \bar{f}(x) = \lim_{n\to\infty} \bar{f}(x_n) = \lim_{n\to\infty} \bar{g}(x_n) = \bar{g}(x) \Rightarrow \bar{f}(x) = \bar{g}(x)\ \forall x \in X$.
- **Existence:**
  claim: $(x_n)_{n=1}^{\infty}$ a seq in S converging to $x \in X \Rightarrow (f(x_n))_{n=1}^{\infty}$ is Cauchy in (Y, d_Y)
  (°°) $\epsilon > 0$ given: f unif cont on $S \Rightarrow \exists \delta > 0$ ST $d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon\ \forall x, y \in S$
  (x_n) converges in $X \Rightarrow (x_n)$ Cauchy in $X \Rightarrow \exists N > 0$ ST $\forall n, m \geq N : d_X(x_n, x_m) < \delta$
  $\Rightarrow d_Y(f(x_n), f(x_m)) < \epsilon$.
- **Define $\bar{f}$:**
  Def: if $x \in X$: let $(x_n) \in S$, $x_n \to x$: then $(f(x_n))_{n=1}^{\infty}$ is Cauchy in Y, and $\lim_{n\to\infty} f(x_n) := \bar{f}(x) \in Y$ as Y is complete
  - **Check:** $\bar{f}$ is well defined: $(x_n), (y_n) \in S$ converge to $x \in X \Rightarrow \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$
    (°°) Define: $\begin{cases} z_{2n-1} = x_n \\ z_{2n} = y_n \end{cases} \Rightarrow (z_n) = (x_1, y_1, x_2, y_2, \dots) \Rightarrow z_n \to x$ and $z_n \in S \Rightarrow \lim_{n\to\infty} f(z_n)$ exists in Y
    $\left.\begin{array}{l} (x_n) \text{ is a subseq of } (z_n) \Rightarrow \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(z_n) \\ (y_n) \text{ is a subseq of } (z_n) \Rightarrow \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(z_n) \end{array}\right\} \Rightarrow \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) \Rightarrow \bar{f}(x) = \bar{f}(x)$ ✓
    So if $x \in S$: take $(x_n)_{n=1}^{\infty} = (x)_{n=1}^{\infty}$ the CST seq: then $f(x_n) = f(x) \Rightarrow \bar{f}(x) = \lim_{n\to\infty} f(x_n) = f(x) \Rightarrow \bar{f} = f$ on S.
  - **Check:** $\bar{f}$ is unif cont on X: $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in X)(d_X(x,y) < \delta \Rightarrow d_Y(\bar{f}(x), \bar{f}(y)) < \epsilon)$.
    (°°) Let $\epsilon > 0$: f unif cont on $S \Rightarrow (\exists \delta > 0)(\forall x, y \in S\ (d_X(x,y) < 2\delta \Rightarrow d_Y(f(x), f(y)) < \frac{\epsilon}{3})$
    - Let $x, y \in X$ with $d_X(x,y) < \delta$: choose seq $(x_n), (y_n) \in S$ with $x_n \to x$, $y_n \to y$
      Then $\exists N' > 0$ ST $\forall n \geq N'$, $d_X(x_n, x) < \frac{\delta}{2}$ and $d_X(y_n, x) < \frac{\delta}{2}$
    - $\lim_{n\to\infty} f(x_n) = \bar{f}(x)$ and $\lim_{n\to\infty} f(y_n) = \bar{f}(y) \Rightarrow \exists N'' > 0$ ST $\forall n \geq N''$: $d_Y(f(x_n), \bar{f}(x)) < \frac{\epsilon}{3}$ and $d_Y(f(y_n), \bar{f}(y)) < \frac{\epsilon}{3}$
    - So $\forall n \geq \max\{N', N''\}$: $d_X(x_n, y_n) \leq d(x_n, x) + d_X(x, y) + d_X(y_n, y) < \frac{\delta}{2} + \delta + \frac{\delta}{2} = 2\delta \Rightarrow d_X(x_n, y_n) < 2\delta$
      So $d_Y(\bar{f}(x), \bar{f}(y)) \leq d_Y(\bar{f}(x), f(x_n)) + d_Y(f(x_n), f(y_n)) + d_Y(\bar{f}(y), f(y_n)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$
      $\Rightarrow d_Y(\bar{f}(x), \bar{f}(y)) < \epsilon$.

$(x_n), (y_n) \in S$
$x, y \in X$

Compact metric Spaces:

① Covers and Compactness:

X a set, $Y \subset X$:

Cover of Y: a family of sets $\{O_\alpha\}_{\alpha\in I}$ ST $Y \subset \bigcup_{\alpha\in I} O_\alpha$

→ a cover $\{O_\alpha\}_{\alpha\in I}$ of Y has a finite subcover if $\exists$ finitely many $\alpha_1,\dots,\alpha_n$ ST $Y \subset O_{\alpha_1}\cup\dots\cup O_{\alpha_n}$

Open cover of Y: X a metric space: the cover $\{O_\alpha\}_{\alpha\in I}$ of Y is an open cover if each O_α is open in X.

Compactness:
- A metric space X is compact if every open cover of X in X has a finite subcover
- A subspace $K\subset X$ is compact if K is compact as a metric subspace.

Prop.: A subset $K\subset X$ is compact (in K) $\iff$ every open cover of K in X has a finite subcover.

Proof: ($\Rightarrow$) if $K\subset X$ is compact, then the metric space (K,d) is compact.
- Let $\{O_\alpha\}_{\alpha\in I}$ be an open cover of K in X: $K\subset\bigcup_{\alpha\in I} O_\alpha$ with O_α open in (X,d)
  So $K=\bigcup_{\alpha\in I}(O_\alpha\cap K)=\bigcup_{\alpha\in I} V_\alpha$ where each V_α is open in (K,d) $\Rightarrow \{V_\alpha\}_{\alpha\in I}$ is an open cover of K in K.
- K is compact $\Rightarrow \{V_\alpha\}_{\alpha\in I}$ has a finite subcover $\Rightarrow K=V_{\alpha_1}\cup\dots\cup V_{\alpha_n}=K\cap(O_{\alpha_1}\cup\dots\cup O_{\alpha_n}) \Rightarrow \boxed{K\subset O_{\alpha_1}\cup\dots\cup O_{\alpha_n}}$

($\Leftarrow$) Every open cover of K in X has a finite subcover:
- Let $\{V_\alpha\}_{\alpha\in I}$ be an open cover of K in (K,d) $\Rightarrow$ each V_α is open in (K,d) and $K=\bigcup_{\alpha\in I} V_\alpha=\bigcup_{\alpha\in I}(K\cap \underset{\text{open in } X}{O_\alpha})=K\cap\left(\bigcup_{\alpha\in I} O_\alpha\right)\subset\bigcup_{\alpha\in I} O_\alpha$
  So $\{O_\alpha\}_{\alpha\in I}$ is an open cover of K in X: pick a finite subcover $K\subset O_{\alpha_1}\cup\dots\cup O_{\alpha_n}$
- Then $K=K\cap(O_{\alpha_1}\cup\dots\cup O_{\alpha_n})=(K\cap O_{\alpha_1})\cup\dots\cup(K\cap O_{\alpha_n})=V_{\alpha_1}\cup\dots\cup V_{\alpha_n} \Rightarrow V_{\alpha_1},\dots,V_{\alpha_n}$ is a finite subcover of K in K.

finite intersect° prop = A family of sets $\{F_\alpha\}_{\alpha\in I}$ has finite intersect° property if: for any finite subfamily $F_{\alpha_1},\dots,F_{\alpha_n}$: $F_{\alpha_1}\cap\dots\cap F_{\alpha_n}\neq\phi$

Prop: A metric space (X,d) is compact ① $\iff$ ② For any family $\{F_\alpha\}_{\alpha\in I}$ of closed sets in X with fin. intersect° prop.: $\bigcap_{\alpha\in I} F_\alpha\neq\phi$.

Proof: ($\Rightarrow$) Argue by contradict°: Suppose X is compact but ② fails
i.e. $\exists$ a family $\{F_\alpha\}_{\alpha\in I}$ of closed sets with finite intersect° prop. ST $\bigcap_{\alpha\in I} F_\alpha=\phi$
- So $\bigcup_{\alpha\in I} F_\alpha^c=X$ with F_α^c is open (as F_α is closed) $\Rightarrow \{F_\alpha^c\}$ is an open cover of X.
- So X compact $\Rightarrow \exists$ a fin. subcover: $X=F_{\alpha_1}^c\cup\dots\cup F_{\alpha_n}^c \Rightarrow \phi=X^c=(F_{\alpha_1}^c)^c\cap\dots\cap(F_{\alpha_n}^c)^c=F_{\alpha_1}\cap\dots\cap F_{\alpha_n}$ ← finite intersect°
  ⇐ since $\{F_\alpha\}$ has finite intersection $\Rightarrow F_{\alpha_1}\cap\dots\cap F_{\alpha_n}\neq\phi$ ...

($\Leftarrow$) Let $\{O_\alpha\}_{\alpha\in I}$ be an open cover of X: $X=\bigcup_{\alpha\in I} O_\alpha \Rightarrow \phi=\bigcap_{\alpha\in I} O_\alpha^c \Rightarrow \{O_\alpha^c\}_{\alpha\in I}$ is a family of closed sets that does not have finite intersect° prop.
Then there are finitely many $\alpha_1,\dots,\alpha_n\in I$ ST $\phi=O_{\alpha_1}^c\cap\dots\cap O_{\alpha_n}^c \Rightarrow X=(O_{\alpha_1}^c)^c\cup\dots\cup(O_{\alpha_n}^c)^c=O_{\alpha_1}\cup\dots\cup O_{\alpha_n}$
$\Rightarrow$ the open cover $\{O_\alpha\}_{\alpha\in I}$ of X has a finite subcover.

① Totally bounded, Sequentially compact sets and Compactness:

X a metric space, $Y \subset X$ a subset:

Totally bounded:
- X is totally bounded if: for any $\varepsilon > 0$, X can be covered by a finite set of balls of radius ε.
  i.e.: $\forall \varepsilon > 0, \exists x_1, \dots, x_n \in X$, ST $X = B(x_1, \varepsilon) \cup \dots \cup B(x_n, \varepsilon)$ (n: depends on ε!)
- $Y \subset X$ is totally bounded if Y is totally bounded as a metric subspace of X.
  i.e.: $\forall \varepsilon > 0, \exists x_1, \dots, x_n \in Y$ ST $Y \subset B(x_1, \varepsilon) \cup \dots \cup B(x_n, \varepsilon)$ (n: depends on ε!)

Prop: X is totally bounded $\Rightarrow$ X is bounded

Proof: take $\varepsilon = 1$: $X = B(x_1, 1) \cup \dots \cup B(x_n, 1)$
if $x, y \in X \Rightarrow x \in B(x_k, 1)$ and $y \in B(x_j, 1) \Rightarrow d(x,y) \leq d(x, x_k) + d(x_k, x_j) + d(x_j, y)$
So $d(x,y) \leq 2 + \max_{1 \leq \ell, \ell' \leq n} d(x_\ell, x_{\ell'}) := M \Rightarrow \forall x, y \in X: d(x,y) \leq M$ for some finite $M > 0 \Rightarrow \operatorname{diam}(X) < \infty$
$\Rightarrow X$ is bounded!

⚠ X bounded $\not\Rightarrow$ X totally bounded

Ex: Normed space $(C([0,1]), \|\cdot\|_2)$, with $\|f\|_2 = \left(\int_0^1 (f(x))^2 dx\right)^{1/2}$ ⤳ closed unit ball: $B = \{f \in C([0,1]) : \|f\|_2 \leq 1\}$
- B is bounded: $(\because)$ $d(f,g) = \|f-g\|_2 \leq \|f\|_2 + \|g\|_2 \leq 1+1 = 2$. $\forall f, g \in B \Rightarrow \operatorname{diam}(B) \leq 2$.
- B is not totally bounded: Let $f_n(x) = \cos(2\pi n x)$, $n = 0, 1, \dots \Rightarrow \|f_n\|_2 \leq 1$ and $\|f_n - f_m\|_2 = 1 \ \forall n \neq m$
  if B was totally bounded: we could find $g_1, \dots, g_\ell$ ST: $B \subset B(g_1, \frac{1}{3}) \cup \dots \cup B(g_\ell, \frac{1}{3})$
  $\{f_n\}_{n=1}^{\infty} \in B$ So $\exists n, m$ ST $n \neq m$ and $f_n, f_m \in B(g_k, \frac{1}{3}) \Rightarrow 1 = \|f_n - f_m\|_2 \leq \|f_n - g_k\|_2 + \|g_k - f_m\|_2 < \frac{1}{3} + \frac{1}{3} = \frac{2}{3} < 1$
  So $1 < 1$ ~~⇔~~ Hence B is NOT totally bounded!

Prop: $(X, \|\cdot\|)$ a normed space:
- If the closed unit ball $B = \{x \in X : \|x\| \leq 1\}$ is totally bounded: then $\dim X < \infty$
- If $\dim X < \infty$, and $Y \subset X$: Y is bounded $\Leftrightarrow$ Y is totally bounded.

  Ex: $(\mathbb{R}^n, \|\cdot\|)$: $Y \subset \mathbb{R}^n$ is bounded $\Leftrightarrow$ Y is totally bounded → prop 13 p 199 of book

Sequential Compactness: (X,d) is sequentially compact if: Any Seq $(x_n)_{n=1}^{\infty}$ in X has a Subseq $(x_{n_k})_{k=1}^{\infty}$ ST $x_{n_k} \to x \in X$ as $k \to \infty$

Thm: (Caracterization of compactness)

Let (X,d) be a metric space: TFAE
(1) X is complete & totally bounded
(2) X is compact
(3) X is sequentially compact

Prop: (X, d) a metric space, $Y \subset X$:
(1) Y is a compact subset of X $\Rightarrow$ Y is closed & bounded
(2) Y is compact $\Leftrightarrow$ Y is closed ⤳ similar to Y is complete $\Leftrightarrow$ Y is closed!

Proof: (1)
- Y is compact $\Rightarrow$ Y is totally bounded $\Rightarrow$ Y is bounded.
- let $x \in cl(Y)$: Let $(x_n)_{n=1}^{\infty}$ be a Seq in Y ST $x_n \to x$: $\lim_{n\to\infty} d(x_n, x) = 0$ with $x \in cl(Y) \Rightarrow \lim_{k\to\infty} (x_{n_k}, x) = 0$
  Y is compact $\Rightarrow$ Y is Seq compact $\Rightarrow$ $\exists$ a subseq $(x_{n_k})_{k=1}^{\infty}$ ST $x_{n_k} \to y \in Y \Rightarrow \lim_{n\to\infty} d(x_{n_k}, y) = 0$
  $\Rightarrow x = y \Rightarrow x \in Y \Rightarrow cl(Y) \subset Y \Rightarrow Y = cl(Y)$ So Y is closed

(2) EXERCISE!

Proof: (of characterizat° of compactness thm)

①⇒②: Goal: "X is complete & totally bounded $\Rightarrow$ X is compact"

Strategy: argue by contradiction $\longrightarrow$ Suppose: "X is complete & totally bounded but **NOT** compact"

so $\exists$ an open cover $\{O_\alpha\}_{\alpha\in I}$ that does not have a finite subcover.

Construct a sequence $F_1, F_2, \dots, F_N, \dots$ of closed sets in X:

Step 1: • X is totally bounded $\Rightarrow$ $\exists$ finitely many pts $x_1^{(1)}, \dots, x_{n_1}^{(1)}$ ST $X = B(x_1^{(1)}, \frac{1}{2}) \cup \dots \cup B(x_{n_1}^{(1)}, \frac{1}{2})$

• X is not compact $\Rightarrow$ $\exists k$ in $1\le k\le n_1$ ST the ball $B(x_k^{(1)}, \frac{1}{2})$ is **NOT** covered by any finite subcollection of $\{O_\alpha\}_{\alpha\in I}$

• Set $F_1 = Cl(B(x_k^{(1)}, \frac{1}{2})) \Rightarrow \operatorname{diam} F_1 \le 1$

Step 2: • X is totally bounded $\Rightarrow$ $\exists$ finitely many pts $x_1^{(2)}, \dots, x_{n_2}^{(2)}$ ST $X = B(x_1^{(2)}, \frac{1}{4}) \cup \dots \cup B(x_{n_2}^{(2)}, \frac{1}{4}) \Rightarrow F_1 = F_1 \cap X = \bigcup_{j=1}^{n_2} F_1 \cap B(x_j^{(2)}, \frac{1}{4})$

• X is not compact $\Rightarrow$ $\exists k$ in $1\le k\le n_2$ ST $F_1 \cap B(x_k^{(2)}, \frac{1}{4})$ is **NOT** covered by any finite subcollect° of $\{O_\alpha\}_{\alpha\in I}$

• Set $F_2 = Cl(F_1 \cap B(x_k^{(2)}, \frac{1}{4}))$: F_1, F_2 closed and $F_1 \supset F_2$ and $\begin{cases}\operatorname{diam}(F_1)\le 1\\ \operatorname{diam}(F_2)\le \frac{1}{2}\end{cases}$ and F_1, F_2 cannot be covered by any finite subcollect° of $\{O_\alpha\}_{\alpha\in I}$

⋮

Step N: Suppose we constructed $F_1, F_2, \dots, F_{N-1}$ closed ST $F_1 \supset F_2 \supset \dots \supset F_{N-1}$, $\operatorname{diam}(F_n) \le \frac{1}{n}$ and each F_n cannot be covered by any finite subcoll of $\{O_\alpha\}_{\alpha\in I}$

• $\exists$ finitely many pts $x_1^{(N)}, \dots, x_{n_N}^{(N)}$ ST $X = B(x_1^{(N)}, \frac{1}{2N}) \cup \dots \cup B(x_{n_N}^{(N)}, \frac{1}{2N})$:

• $\exists k$ in $1\le k\le n_N$ ST $F_{N-1} \cap B(x_k^{(N)}, \frac{1}{2N})$ cannot be covered by any subcoll° of $\{O_\alpha\}$ (as F_{N-1} cannot)

• Set $F_N = Cl(F_{N-1} \cap B(x_k^{(N)}, \frac{1}{2N}))$: F_N closed, $F_{N-1} \supset F_N$, F_N cannot be covered by any fin. subcoll of $\{O_\alpha\}_{\alpha\in I}$, $\operatorname{diam}(F_N) \le \frac{1}{N}$.

⋮

So: • (X,d) is complete: Apply Cantor's intersect° thm to $\{F_N\}_{N=1}^{\infty}$ $\Rightarrow$ $\exists x_0 \in X$ ST: $\bigcap_{N=1}^{\infty} F_N = \{x_0\}$

• $\{O_\alpha\}_{\alpha\in I}$ covers X $\Rightarrow$ $\exists \alpha_0 \in I$ ST $x_0 \in O_{\alpha_0}$

• O_{α_0} is open $\Rightarrow$ $\exists r>0$ ST $B(x_0, r) \subset O_{\alpha_0}$ $\Big\}$ Choose N_0 ST $\frac{1}{N_0} < r$: then $x_0 \in F_{N_0}$ and $\operatorname{diam}(F_{N_0}) < \frac{1}{N_0}$

• If $y \in F_{N_0}$: $d(x_0, y) \le \operatorname{diam}(F_{N_0}) < \frac{1}{N_0} < r \Rightarrow F_{N_0} \subset B(x_0, r) \subset O_{\alpha_0} \Rightarrow F_{N_0}$ is covered by O_{α_0}, so F_{N_0} has a finite subcover in $\{O_\alpha\}_{\alpha\in I}$ $\boxed{\Rightarrow\Leftarrow}$

②⇒③: Goal: "X is compact $\Rightarrow$ X is seq compact"

• Let $(x_n)_{n=1}^{\infty}$ be a seq in X: $F_n := Cl(\{x_k : k \ge n\})$ So F_n is closed and $F_{n+1} \subset F_n$, and $x_\ell \in F_{n_1} \cap F_{n_2} \cap \dots \cap F_{n_k}$ where $\ell = \max\{n_1, \dots, n_k\}$

• X is compact $\Rightarrow \bigcap_{n=1}^{\infty} F_n \ne \phi \Rightarrow$ pick $x_0 \in \bigcap_{n=1}^{\infty} F_n$:

Step 1: $x_0 \in F_1 = Cl\{x_k : k\ge 1\} \Rightarrow x_0$ is a closure pt $\Rightarrow B(x_0, 1) \cap \{x_k : k \ge 1\} \ne \phi$: let $x_{n_1} \in B(x_0, 1) \cap \{x_k : k\ge 1\} \Rightarrow d(x_{n_1}, x_0) < 1$

Step 2: So $x_0 \in F_{n_1+1} = Cl\{x_k : k \ge n_1+1\} \Rightarrow B(x_0, \frac{1}{2}) \cap \{x_k : k \ge n_1+1\} \ne \phi$; let $x_{n_2} \in B(x_0, \frac{1}{2}) \cap \{x_k : k \ge n_1+1\} \Rightarrow n_2 > n_1$, $d(x_{n_2}, x_0) < \frac{1}{2}$

⋮

Step N: Pick $x_{n_k} \in F_{n_{k-1}+1}$ ST $d(x_{n_k}, x_0) < \frac{1}{k}$

• So: (x_{n_k}) is a subseq of (x_n) ST $d(x_{n_k}, x_0) < \frac{1}{k} \Rightarrow \lim_{k\to\infty} d(x_{n_k}, x_0) = 0 \Rightarrow \lim_{k\to\infty} x_{n_k} = x_0 \in X$.

③⇒①: Goal: "X is seq compact $\Rightarrow$ X is complete & totally bounded"

• Let $(x_n)_{n=1}^{\infty}$ be Cauchy in X: X is seq compact $\Rightarrow (x_n)_{n=1}^{\infty}$ has a convergent subseq $(x_{n_k})_{k=1}^{\infty}$ ST $x_{n_k} \to x \in X$ as $k \to \infty$.

• So $(x_n)_{n=1}^{\infty}$ is a Cauchy seq with a convergent subseq $(x_{n_k})_{k=1}^{\infty} \Rightarrow (x_n)_{n=1}^{\infty}$ converge to $x \in X$ $\Rightarrow X$ is complete!

• Suppose X is not totally bounded: $\exists \varepsilon_0 > 0$ ST X is **NOT** covered by a finite collect° of balls of radius $\varepsilon_0 > 0$.

Step 1: pick $x_1 \in X$: then $X \ne B(x_1, \varepsilon_0)$

Step 2: So $\exists x_2 \notin B(x_1, \varepsilon_0)$ ST $d(x_2, x_1) \ge \varepsilon_0 \Rightarrow X \ne B(x_1, \varepsilon_0) \cup B(x_2, \varepsilon_0)$

Step 3: So $\exists x_3 \notin B(x_1, \varepsilon_0) \cup B(x_2, \varepsilon_0)$ ST $\begin{cases} d(x_3, x_1) \ge \varepsilon_0 \\ d(x_3, x_2) \ge \varepsilon_0 \end{cases} \Rightarrow X \ne B(x_1, \varepsilon_0) \cup B(x_2, \varepsilon_0) \cup B(x_3, \varepsilon_0)$

⋮

Step N: $X \ne B(x_1, \varepsilon_0) \cup \dots \cup B(x_N, \varepsilon_0)$

So: We constructed a seq (x_n) in X ST $d(x_n, x_m) \ge \varepsilon_0$ $\forall n \ne m$

• X seq compact $\Rightarrow (x_n)$ has a convergent subseq $(x_{n_k})_{k=1}^{\infty}$ in X

• **BUT**: $d(x_{n_k}, x_{n_j}) \ge \varepsilon_0$ $\forall k \ne j \Rightarrow (x_{n_k})$ is **NOT** Cauchy $\Rightarrow (x_{n_k})$ is **NOT** convergent in X $\Rightarrow\Leftarrow$.

III Compactness & Continuous functions:

Prop: (X, d_X), (Y, d_Y) metric spaces, $f: X \to Y$ a continuous function:
X is compact $\Rightarrow$ $f(X) = \{f(x) : x \in X\}$ is a compact subset of Y.

Proof: Let $\{O_\alpha\}_{\alpha \in I}$ be an open cover of $f(X)$: $f(X) \subset \bigcup_{\alpha \in I} O_\alpha \Rightarrow X \subset f^{-1}\left(\bigcup_{\alpha \in I} O_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(O_\alpha)$ WHY?

- Since $f: X \to Y$, then $X = \bigcup_{\alpha \in I} f^{-1}(O_\alpha)$
- f cont: so O_α is open in Y $\Rightarrow f^{-1}(O_\alpha)$ is open in X $\Rightarrow \{f^{-1}(O_\alpha)\}_{\alpha \in I}$ is an open cover of X.
- X is compact: so this cover has a finite subcover: $\exists \alpha_1, \dots, \alpha_N \in I$ ST $X = f^{-1}(O_{\alpha_1}) \cup \dots \cup f^{-1}(O_{\alpha_N})$

$\Rightarrow f(X) = f\left(\bigcup_{k=1}^N f^{-1}(O_{\alpha_k})\right) = \bigcup_{k=1}^N f(f^{-1}(O_{\alpha_k})) \subset \bigcup_{k=1}^N O_{\alpha_k}$ (WHY?)

$\Rightarrow$ the open cover $\{O_\alpha\}_{\alpha \in I}$ of $f(X)$ has a finite subcover $\Rightarrow f(X)$ is compact

Exercise: Prove this using seq. compactness

thm: (Extreme Value theorem)

(X, d) a compact metric space, $f: X \to \mathbb{R}$ a cont funct $\Rightarrow$ f achieves its min and max on X.

Note: Converse statement in book

Proof: X is compact $\Rightarrow f(X)$ is a compact subset of $\mathbb{R}$ $\Rightarrow f(X)$ is closed & bounded $\Rightarrow \begin{cases} M = \sup(f(X)) = \sup_{x \in X} f(x), & |M| < \infty \\ m = \inf(f(X)) = \inf_{x \in X} f(x), & |m| < \infty \end{cases}$ since bounded set

Claim: $\exists x_0, y_0 \in X$ ST $f(x_0) = M$ and $f(y_0) = m$, so $f(x_0) \geq f(x)\ \forall x \in X$ and f achieves its max value at x_0 $\leadsto$ similar for y_0

Case 1: deal with M:

- Suppose NOT: i.e. $\forall x \in X$, $f(x) < M = \sup\{f(x) : x \in X\}$ $\Rightarrow \forall n > 0$, $M - \frac{1}{n}$ is not a UB for $f(x)$ since M is the least UB.
- So $\exists x_n \in X$ ST $f(x_n) > M - \frac{1}{n}$: X is compact $\Rightarrow X$ is seq compact $\Rightarrow \exists$ a subseq $(x_{n_k}) \subset X$ ST $\lim_{k \to \infty} x_{n_k} = x_0$
- f cont $\Rightarrow \lim_{k \to \infty} f(x_{n_k}) = f(x_0)$
- But $M > f(x_{n_k}) > M - \frac{1}{n_k} \Rightarrow \lim_{k \to \infty} f(x_{n_k}) = M = f(x_0)$ since $k \to \infty \Rightarrow n \to \infty$. $\Rightarrow$ f achieves its max value at x_0.

Case 2: for m: take $f \to -f$: then $(-f)$ attains its max on X $\Rightarrow \exists m \in \mathbb{R}$ ST $-m \geq -f(x)$ $\forall x \in X \Rightarrow f(x) \geq m\ \forall x \in X$.

Ex: $f(x) = \frac{1}{x}$ on $(0, \infty)$: $\begin{cases} M = \sup(f(X)) = \infty \\ m = \inf(f(X)) = 0 \end{cases}$ but f never achieves these value. ($\because$) $X = (0, \infty)$ is NOT compact

IV Lebesgue Number:

thm: (Lebesgue Converging lemma)

Let (X, d) be a compact metric space: Let $\{O_\alpha\}_{\alpha \in I}$ be an open cover of X:
Then $\exists \varepsilon > 0$ ST $\forall x \in X$, $B(x, \varepsilon)$ is contained in some element O_α of the cover $\to$ Note: ε does NOT depend on x!

Note: Such an ε is called a Lebesgue number for the cover $\{O_\alpha\}_{\alpha \in I}$. ε is not unique: every $\varepsilon' < \varepsilon$ also works!

Proof: By contradiction: suppose that "$(\exists \varepsilon > 0)(\forall x \in X)(\exists \alpha \in I)(B(x, \varepsilon) \subset O_\alpha)$" is NOT true.
so $(\forall \varepsilon > 0)(\exists x \in X)(\forall \alpha \in I)(B(x, \varepsilon) \not\subset O_\alpha)$

- Take $\varepsilon = \frac{1}{n}$: then $\exists x_n \in X$ ST $B(x_n, \frac{1}{n})$ is NOT contained in any single member of the cover
- X is compact $\Rightarrow X$ is seq compact $\Rightarrow (x_n)_{n=1}^\infty$ has a convergent subsequence $(x_{n_k})_{k=1}^\infty$ ST $\lim_{k \to \infty} x_{n_k} = x_0 \in X$.
- So $\exists x_0$ ST $x_0 \in O_{\alpha_0}$, and $\exists r > 0$ ST $B(x_0, r) \subset O_{\alpha_0}$ because O_{α_0} is open!
- choose n_k ST $d(x_0, x_{n_k}) < \frac{r}{2}$ and $\frac{1}{n_k} < \frac{r}{2}$ ($\because \lim_{k \to \infty} d(x_0, x_{n_k}) = 0$ since $k \to \infty \Rightarrow n \to \infty$)
- Look at $B(x_n, \frac{1}{n_k})$: if $x \in B(x_n, \frac{1}{n_k}) \Rightarrow d(x_0, x) \leq d(x_0, x_{n_k}) + d(x_{n_k}, x) < \frac{r}{2} + \frac{1}{n_k} < \frac{r}{2} + \frac{r}{2} = r$.

So $B(x_{n_k}, \frac{1}{n_k}) \subset B(x_0, r) \subset O_{\alpha_0}$ $\Rightarrow B(x_{n_k}, \frac{1}{n_k})$ is contained in a single element O_{α_0} of the cover $\{O_\alpha\}_{\alpha \in I}$! $\Rightarrow\Leftarrow$

Thm : (X, d_X), (Y, d_Y) metric spaces, $f : X \to Y$ a continuous function.

X is compact $\Rightarrow$ f is uniformly continuous on X

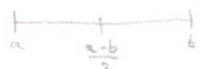
Proof : Goal : " $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in X)(d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon)$ "

- f is continuous $\Rightarrow$ $\forall \epsilon > 0$ fixed : $\forall x \in X$, $\exists \delta > 0$ st $d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \frac{\epsilon}{2}$.
- Let $O_x = B(x, \delta_x)$: $\{O_x\}_{x \in X}$ is an open cover of X, so $X = \bigcup_{x \in X} O_x$ (here the index set is the whole set: $I = X$).
- If $u, v \in O_x = B(x, \delta_x)$: $d_Y(f(u), f(v)) \leq d_Y(f(u), f(x)) + d_Y(f(x), f(v)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.
- X is compact $\Rightarrow$ $\exists$ a Lebesgue nb $\delta > 0$ of the cover $\{O_x\}_{x \in X}$: So given $\epsilon > 0$, if $x, y \in X$ and $d_X(x,y) < \delta$, then $y \in B(x, \delta)$

Hence $\exists x' \in X$ st $B(x, \delta) \subset O_{x'}$ $\Rightarrow$ $\forall x, y \in O_{x'}$: $d_Y(f(x), f(y)) < \epsilon$.

Prop : A closed interval $[a,b]$ of $\mathbb{R}$ is a compact subset of $\mathbb{R}$. $\rightsquigarrow$ Any Seq in $[a,b]$ has a convergent subseq

Proof :



step 1 : divide $[a,b]$ into 2 parts of equal length : call I_1 one of these intervals containing ∞ many elements of the Seq

Let x_{n_1} be the 1st element of the Seq in I_1.

step 2 : divide I_1 into 2 parts of equal length : call I_2 one of these intervals containing ∞ many elements of the Subseq

Let x_{n_2} be the 1st element of I_2 (with $n_2 > n_1$) : also, $x_{n_2} \in I_1$.

continue inductively : construct a seq of intervals I_k of length $\frac{b-a}{2^k}$, and a subseq $(x_{n_k})_{k=1}^{\infty}$ st $x_{n_j} \in I_k$ $\forall j \geq k$.

So : • $(x_{n_k})_{k=1}^{\infty}$ is Cauchy because : $x_{n_j}, x_{n_{j'}} \in I_k$ for $j, j' \geq k$ $\Rightarrow$ $|x_{n_j} - x_{n_{j'}}| \leq \frac{b-a}{2^k} < \epsilon$ for large enough k given $\epsilon > 0$.

• $\mathbb{R}$ is complete $\Rightarrow$ $\lim_{k \to \infty} x_{n_k} = x \in \mathbb{R}$ exists, and $a \leq x \leq b$ because $a \leq x_{n_k} \leq b$ $\forall k > 0$.

Prop : X a nonempty set, d the discrete metric on X : $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$

X is compact $\Leftrightarrow$ X is a finite set

Proof : $\Leftarrow$ Any metric space consisting of finitely many pts is compact.

$\Rightarrow$ (X, d) is compact : $B(x, \frac{1}{2}) = \{x\}$ $\Rightarrow$ $X = \bigcup_{x \in X} \{x\}$ is an open cover of X as $\{x\}$ is an open set.

So by compactness, $\exists$ a finite subcover : $\exists x_1, \dots, x_n \in X$ st $X = \{x_1\} \cup \dots \cup \{x_n\}$ $\Rightarrow$ $X = \{x_1, \dots, x_n\}$ is a finite set !

V) Compactness in finite dimensional normed space :

Lemma : X a finite dimensional Vspace of dimension $\dim X = N < \infty$.
Let $\{e_1, \dots, e_N\}$ be a basis for X : if $x \in X$, $x = \alpha_1 e_1 + \dots + \alpha_N e_N$ for some $\alpha_i \in \mathbb{R}$.
then $\|x\|_\infty = \max_{1 \le j \le N} |\alpha_j|$ is a norm on X $\longrightarrow$ NOTE : $\|\cdot\|_\infty$ depends on the choice of basis!

Proof : • $\|x\|_\infty \geq 0$ ✓, $\|x\|_\infty = 0 \Leftrightarrow \max|\alpha_j| = 0 \Leftrightarrow \alpha_j = 0$ ✓, $\|\lambda \cdot x\|_\infty = \max|\lambda \cdot \alpha_j| = |\lambda| \cdot \|x\|_\infty$ ✓
• $\|x+y\|_\infty = \max|\alpha_j + \alpha_j'| \leq \max(|\alpha_j| + |\alpha_j'|) = \|x\|_\infty + \|y\|_\infty$

Prop : Let $S \subset X$: S is closed & bounded in $(X, \|\cdot\|_\infty) \Longleftrightarrow S$ is compact in $(X, \|\cdot\|_\infty)$

Proof : $\Leftarrow$ True for any metric space
$\Rightarrow$ Goal : Show that S is sequentially compact.
• Let (x_n) be a seq in S : $x_n = \alpha_1^{(n)} e_1 + \dots + \alpha_N^{(n)} e_N \longrightarrow \|x_n\|_\infty = \max_{1 \le j \le N} |\alpha_j^{(n)}|$
• S is bounded $\Rightarrow \exists M > 0$ ST $\|x_n\|_\infty \leq M$ $\forall n \in \mathbb{N} \Rightarrow |\alpha_j^{(n)}| \leq M$ $\forall j, n \Rightarrow \forall j$: $(\alpha_j^{(n)})_{n=1}^\infty$ is a bounded seq of real nb
So $(\alpha_j^{(n)})_{n=1}^\infty$ has a convergent subsequence :
• $\alpha_1^{(n_{11})}, \alpha_1^{(n_{21})}, \dots, \alpha_1^{(n_{k1})}, \dots, \lim_{k\to\infty} \alpha_1^{(n_{k1})}$ exists : look at $\alpha_2^{(n)}$ among the $\alpha_1^{(n_{k1})}$ (this forms a bounded seq) and extract a converging subseq $(\alpha_2^{(n_{k2})})$
• $\alpha_2^{(n_{12})}, \alpha_2^{(n_{22})}, \dots, \alpha_2^{(n_{k2})}, \dots, \lim_{k\to\infty} \alpha_2^{(n_{k2})}$ exists : is also a bounded seq look at $\alpha_3^{(n)}$ among $\alpha_2^{(n_{k2})} \to$ extract a conv subseq $(\alpha_3^{(n_{k3})})$

So :
$n_{11}, n_{21}, \dots, n_{k1}, \dots, \infty$
$n_{12}, n_{22}, \dots, n_{k2}, \dots, \infty$
subseq
$n_{1N}, n_{2N}, \dots, n_{kN}, \dots, \infty$

seq of indices $(n_{kN})_{k=1}^\infty$, ST $\forall 1 \leq j \leq N$: $\lim_{k\to\infty} \alpha_j^{(n_{kN})} := \alpha_j$ exists
Let $x_{n_k} = \sum_{j=1}^N \alpha_j^{(n_{kN})} \Rightarrow (x_{n_k})_{k=1}^\infty$ is a subseq of (x_n)
Let $x = \sum_{j=1}^N \alpha_j e_j$

then : $\|x_{n_k} - x\|_\infty = \|\sum_{j=1}^N (\alpha_j^{(n_{kN})} - \alpha_j) e_j\|_\infty = \max_{1\le j\le N} |\alpha_j^{(n_{kN})} - \alpha_j| \leq \sum_{j=1}^N |\alpha_j^{(n_{kN})} - \alpha_j| \to 0$ as $k \to \infty$
• S is closed $\Rightarrow x \in S$

conclusion : Any seq $(x_n)_{n=1}^\infty \in S$ has a subsequence $(x_{n_k})_{k=1}^\infty$ that converges to an element $x \in S$.

Thm : X a finite dimensional Vspace : Any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are equivalent.

Proof : Goal : $\exists c_1, c_2 > 0$ ST $c_1\|x\|_2 \leq \|x\|_1 \leq c_2\|x\|_2$ $\forall x \in X$
• Fix a basis $\{e_1, \dots, e_N\}$ of X : let $\|\cdot\|_\infty$ be as in the previous prop, and let $\|\cdot\|$ be any other norm on X
claim : $\|\cdot\| \sim \|\cdot\|_\infty$ are equivalent
(a) • $\|x\| \leq c_2 \cdot \|x\|_\infty$:
$x = \alpha_1 e_1 + \dots + \alpha_N e_N \Rightarrow \|x\| = \|\sum_{j=1}^N \alpha_j e_j\| \leq \sum_{j=1}^N \|\alpha_j e_j\| = \sum_{j=1}^N |\alpha_j| \cdot \|e_j\| \leq (\max_{1\le j\le N}|\alpha_j|) \cdot \sum_{j=1}^N \|e_j\| = \|x\|_\infty \cdot \sum_{j=1}^N \|e_j\|$
So $c_2 = \sum_{j=1}^N \|e_j\| < \infty$ depends only on $\|\cdot\|$ and the choice of basis $\Rightarrow \|x\| \leq c_2\|x\|_\infty$ $\forall x \in X$
• $c_1\|x\|_\infty \leq \|x\|$: $(X, \|\cdot\|_\infty) \to (\mathbb{R}, |\cdot|)$
Define a functᵒ : $f : X \to \mathbb{R}$ by $f(x) = \|x\|$: $|f(x) - f(y)| = |\|x\| - \|y\|| \leq \|x - y\| \leq c_2\|x - y\|_\infty \Rightarrow f$ Lipschitz on $(X, \|\cdot\|_\infty)$ $\Rightarrow f$ cont on $(X, \|\cdot\|_\infty)$
$S = \{x \in S : \|x\|_\infty = 1\}$ is the unit sphere in $(X, \|\cdot\|_\infty) \to S$ is bounded in $(X, \|\cdot\|_\infty)$
S is closed & bounded in $(X, \|\cdot\|_\infty)$: (a) $x_n \in S$, $x_n \to x \Rightarrow |1 - \|x\|_\infty| = |\|x_n\|_\infty - \|x\|_\infty| \leq \|x_n - x\|_\infty < \varepsilon \Rightarrow \|x\|_\infty = 1 \Rightarrow x \in S$ $\Rightarrow S$ is closed
$\Rightarrow S$ is compact in $(X, \|\cdot\|_\infty)$ by previous prop, and $f(x) = \|x\|$ is cont on the restrictᵒ of any metric subspace of $(X, \|\cdot\|_\infty)$
$\Rightarrow f$ restricted to S is continuous, S is compact $\Rightarrow f$ achieves a minimum value on S
So $\exists x_0 \in S$ ST $m = \|x_0\| = \inf_{x \in S}\{\|x\|\}$ $\to$ must have $m > 0$ otherwise if $m = 0$: $x_0 = 0_X$, and $x_0 \in S \Rightarrow \|x_0\| = 0, \|x_0\|_\infty = 1 \neq 0$ ($\Rightarrow \Leftarrow$)
take any nonzero $x \in X$: then $\frac{x}{\|x\|_\infty} \in S$ as $\|\frac{x}{\|x\|_\infty}\|_\infty = 1$, So $\|\frac{x}{\|x\|_\infty}\| \geq m \Leftrightarrow \|x\| \geq m\|x\|_\infty$ $\forall x \in X, x \neq 0_X$.
This is still valid for $x = 0_X \Rightarrow c_1 = m > 0$: $c_1\|x\|_\infty \leq \|x\|$ $\forall x \in X$
conclusion : So any norm $\|\cdot\|_1 \sim \|\cdot\|_\infty$, $\|\cdot\|_2 \sim \|\cdot\|_\infty \Rightarrow \|\cdot\|_1 \sim \|\cdot\|_2$

Thm : (Heine - Borel) $(X, \|\cdot\|)$ a fin. dim Vspace, $S \subset X$: S is closed & bounded in $(X, \|\cdot\|) \Leftrightarrow S$ is compact in $(X, \|\cdot\|)$

Proof : $\Leftarrow$ Always True (general fact)
$\Rightarrow$ Fix a basis $\{e_1, \dots, e_N\}$: then $\|\cdot\| \sim \|\cdot\|_\infty \Rightarrow \exists c_1, c_2 > 0$ st $c_1\|x\| \leq \|x\|_\infty \leq c_2\|x\|$ $\forall x \in X$
• S closed & bounded in $(X, \|\cdot\|) \Rightarrow S$ is bounded in $(X, \|\cdot\|_\infty)$ by 1st ineq ⊕ S is closed in $(X, \|\cdot\|_\infty)$ by 2nd ineq ($x \in Cl(S) \Leftrightarrow \exists (x_n) \in S$ ST $\|x_n - x\|_\infty \to 0$ in $(X, \|\cdot\|_\infty)$ $\Rightarrow \|x_n - x\| \leq \frac{1}{c_1}\|x_n - x\|_\infty \to 0$ as S closed in $(X, \|\cdot\|)$ $\Rightarrow \|x_n - x\| \to 0 \Rightarrow x \in S$)
$\Rightarrow$ So S is compact in $(X, \|\cdot\|_\infty)$ by previous prop
• (x_n) a seq in S : S compact in $(X, \|\cdot\|_\infty) \Rightarrow (x_n)$ has a convergent subseq $(x_{n_k})_{k=1}^\infty$: $x_{n_k} \to x \in S$ in $(X, \|\cdot\|_\infty)$
, but $\|\cdot\| \sim \|\cdot\|_\infty \Rightarrow x_{n_k} \to x$ in $(X, \|\cdot\|)$ too! $\Rightarrow S$ is seq compact in $(X, \|\cdot\|) \Rightarrow S$ is compact in $(X, \|\cdot\|)$

VI Compactness in ∞ dimensional normed spaces :

Lemma 1 = (Reisz's Lemma) $(X, \|\cdot\|)$ a normed space, $S \subset X$ a <u>proper</u> closed subspace of X. Let $0<\alpha<1$: then $\exists y_\alpha \in X \setminus S$ ST $\|y_\alpha - x\| > \alpha$ and $\|y_\alpha\| = 1$ $\forall x \in S$

Ex : $\mathbb{R}^2$

Proof : S is a proper subset of $X \Rightarrow \exists y \in X \setminus S$: let $d = dist(y, S) = \inf\{\|y-x\| : x \in S\}$

- $d>0$:
  (ep) if $d=0$: $\forall n>0$, $\frac{1}{n}$ is not a LB for our set $\Rightarrow \exists x_n \in S$ ST $\|y - x_n\| < \frac{1}{n} \Rightarrow x_n \to y$ and $y \in S$ because S is closed $\Rightarrow \Leftarrow$ as $y \notin S$
- For $0<\alpha<1$: $\frac{d}{\alpha} > d \Rightarrow \frac{d}{\alpha}$ is not a LB for $\{\|y-x\|, x \in S\} \Rightarrow \exists x_\alpha \in S$ ST $\underline{\|y - x_\alpha\| < \frac{d}{\alpha}}$.

  Let $y_\alpha = \frac{y - x_\alpha}{\|y - x_\alpha\|} \Rightarrow \boxed{\|y_\alpha\| = 1}$ and $y_\alpha \notin S$ because if $y_\alpha \in S \Rightarrow y = \|y - x_\alpha\| y_\alpha + x_\alpha \in S$ because $x_\alpha \in S$ and S is a Vspace (So any lin comb $\in S$)

  Let $x \in S$: $\|y_\alpha - x\| = \left\|\frac{y - x_\alpha}{\|y - x_\alpha\|} - x\right\| = \frac{1}{\|y - x_\alpha\|} \cdot \|y - \underbrace{(\overset{\in S}{x_\alpha} + \|y - x_\alpha\| \overset{\in S}{x})}_{\in S}\| \geqslant \frac{1}{\|y - x_\alpha\|} d > \frac{\alpha}{d} \cdot d = \alpha$ as $\|y - x_\alpha\| < \frac{d}{\alpha} \Rightarrow \|y_\alpha - x\| > \alpha$ $\forall x \in S$

Lemma 2 = $(X, \|\cdot\|)$ a normed space, $S \subset X$ a <u>finite</u> dim. Subspace $\Longrightarrow$ S is Closed.

Proof = • Let $x \in Cl(S) \Rightarrow \exists$ a Seq $(x_n)_{n=1}^\infty$ in S ST $\lim_{n\to\infty} \|x_n - x\| = 0 \Rightarrow (x_n)$ is Convergent $\Rightarrow (x_n)$ is bounded : $\exists M>0$ ST $\|x_n\| \leq M$ $\forall n \in \mathbb{N}$

- $(S, \|\cdot\|)$ is a fin. dim normed subspace of X, and $(x_n) \in$ Closed bal $\{x \in S : \|x\| \leq M\}$ is a closed & bounded subset of S
- $(S, \|\cdot\|)$ finite dim $\Rightarrow$ the closed ball is compact ($\Leftarrow$ closed & bounded) $\Rightarrow \exists$ a Convergent Subseq (x_{n_k}) converging to $y \in S$
- So $\lim_{k\to\infty} \|x_{n_k} - y\| = 0$ and $\lim_{n\to\infty} \|x_n - x\| = 0 \Rightarrow x = y \Rightarrow x \in S \Rightarrow Cl(S) \subset S \Rightarrow S$ is closed

Thm : $(X, \|\cdot\|)$ a normed Space : $\dim X = \infty \Longrightarrow$ the closed unit ball $B = \{x \in X : \|x\| \leq 1\}$ is <u>NOT</u> a compact Subset of X.

Proof : • take $x_1 \in B$ ST $\|x_1\| = 1$: let $S_1 = Span\{x_1\} = \{\alpha_1 x_1 : \alpha_1 \in \mathbb{R}\}$: $\dim S = 1 < \infty \Rightarrow S_1$ is a closed Vspace of X and $S_1 \neq X$ ($\dim S_1 < \dim X = \infty$)

- So $\exists x_2 \in X \setminus S_1$ ST $\|x_2\| = 1$ and $\|x_2 - x_1\| \geqslant \frac{1}{2}$ by Reisz : let $S_2 = Span\{x_1, x_2\} = \{\alpha_1 x_1 + \alpha_2 x_2 : \alpha_1, \alpha_2 \in \mathbb{R}\} \Rightarrow S_2$ is a closed Vspace of X and $S_2 \neq X$.
- So $\exists x_3 \in X \setminus (S_1 \cup S_2)$ ST $\|x_3\| = 1$, $\|x_3 - x_1\| \geqslant \frac{1}{2}$ and $\|x_3 - x_2\| \geqslant \frac{1}{2}$ by Reisz : $S_3 = Span\{x_1, x_2, x_3\} \Rightarrow S_3$ is a closed Vspace of X, $S_3 \neq X$
- We construct inductively a Seq $(x_n)_{n=1}^\infty$ ST $\|x_n\| = 1$, $\|x_n - x_m\| \geqslant \frac{1}{2}$ $\forall n \neq m$
- So this seq is <u>NOT</u> Cauchy and cannot have a Cauchy subseq $\Rightarrow$ No Subseq can Converge $\Rightarrow$ the Seq does not converge $\Rightarrow S$ Not Seq compact $\Rightarrow S$ not Compact

Note : $(X, \|\cdot\|)$ a normed space : the unit ball is compact $\Longleftrightarrow \dim X < \infty$

Product of compact metric spaces =

Thm : (1) Let $(X_1, d_1), \dots, (X_n, d_n)$ be <u>finitely</u> many compact metric spaces $\Rightarrow$ their product (X, d) is compact.

(2) Let (X_k, d_k), $k = 1, 2, \dots$ be a seq of compact metric spaces $\Rightarrow$ their product (X, d) is compact.

Proof : (2) $X = \{x = (x_k)_{k=1}^\infty : x_k \in X_k\}$, $d(x,y) = \sum_{k=1}^\infty 2^{-k} \frac{d_k(x,y)}{1 + d_k(x,y)}$

<u>Goal</u> = A given Seq $(x^{(n)})_{n=1}^\infty$ in (X, d) has a convergent subseq $\Longrightarrow$ We will Use Cantor's diagonal argument!

- Convergence in the product space is equivalent to componentwise convergence : <u>Goal</u> : $\exists$ a subseq $x^{(n_{jj})} = (x_k^{(n_{jj})})_{k=1}^\infty$ ST $\lim_{j\to\infty} x_k^{(n_{jj})}$ exists $\forall k$

<u>Step 1</u> : $(x_1^{(n)})_{n=1}^\infty$ is a seq of the 1st components : X_1 compact $\Rightarrow \exists$ a Converging subseq : $x_1^{(n_{11})}, x_1^{(n_{21})}, \dots, x_1^{(n_{j1})}, \dots$

<u>Step 2</u> : look at X_2 along $n_{11}, n_{21}, \dots, n_{j1}$: $x_2^{n_{11}}, x_2^{n_{21}}, \dots, x_2^{n_{j1}}, \dots \Rightarrow$ Extract a conv subseq : $x_2^{(n_{12})}, x_2^{(n_{22})}, \dots, x_2^{(n_{j2})}, \dots$

<u>Step j</u> : Suppose that $x_{j-1}^{(n_{1,j-1})}, x_{j-1}^{(n_{2,j-1})}, \dots, x_{j-1}^{(n_{j-1,j-1})}, \dots$ was constructed $\to$ look at x_j along $n_{1,j-1}, n_{2,j-1}, \dots n_{j,j-1}, \dots$: $x_j^{n_{1,j-1}}, x_j^{n_{2,j-1}}, \dots, x_j^{n_{j,j-1}}, \dots$

$\to$ Pick a conv subseq : $x_j^{(n_{j,1})}, x_j^{(n_{j,2})}, \dots, x_j^{(n_{j,j})}, \dots$

<u>You get</u> :

$$\begin{array}{ccccc} n_{11} & n_{21} & n_{31} & \cdots & n_{j1} \cdots \\ n_{12} & n_{22} & n_{32} & \cdots & n_{j2} \cdots \\ n_{13} & n_{23} & n_{33} & \cdots & n_{j3} \cdots \\ \vdots & & & & \\ n_{1j} & n_{2j} & n_{3j} & \cdots & n_{jj} \\ \vdots & \vdots & & & \end{array}$$

All subseq of the previous

$\to$ this procedure does not stop (∞ prod $\Rightarrow$ no last term!)

So pick a subseq $x^{(n_{jj})} = (x_k^{(n_{jj})})_{k=1}^\infty$, $j = 1, 2, \dots \Rightarrow$ diagonal elements!

• <u>Diagonal argument</u> : $\forall k$, the limit $\lim_{j\to\infty} x_k^{(n_{jj})}$ exists! (i.e. the Subseq Converges)

(eg) $x_1^{(n_{jj})}$, $j = 1, 2, \dots$ is a Subseq of $x_1^{(n_{j1})}$ so it converges.

$x_2^{(n_{jj})}$, $j = 1, 2, \dots$ ———— $x_2^{(n_{j2})}$ except possibly for 1st element so it converges.

$\forall k$: $x_k^{(n_{jj})}$, $j = 1, 2, \dots$ ———— $x_k^{(n_{jk})}$ ———— $(k-1)$ first elements so it converge.

Separable Metric Spaces :

(I) Countable & Uncountable Sets :

Countable set : • A set S is countable if $\exists$ a surjection $f : \mathbb{N} \to S$: so the set S can be enumerated as : $S = \{s_1, s_2, \dots\}$ where $f(k) = s_k$ (S could be a finite set).
• if S is ∞ then $\exists$ a bijection $f : \mathbb{N} \longleftrightarrow S$

Uncountable set : if S is not countable, S is uncountable.

Prop : • $S' \subset S$: S countable $\Rightarrow$ S' countable.
• $S_1, \dots, S_n$ countable sets (n is a finite nb) $\Longrightarrow$ Cartesian prod $S = S_1 \times \cdots \times S_n = \{s = \{s_1, \dots, s_n\} : s_k \in S_k\}$ is countable.
• A countable union of countable sets is countable : S_i countable, I countable $\Rightarrow \bigcup_{i \in I} S_i$ is countable.

Corr : • $\mathbb{Z}$ is countable. ($\because$ $f : \mathbb{N} \to \mathbb{Z}$: $\begin{cases} f(2n) = n \\ f(2n-1) = -n+1 \end{cases}$ $\Rightarrow \{1, 2, 3, 4, \dots\} \to \{0, 1, -1, 2, -2, \dots\}$)
• $\mathbb{Q}$ is countable. ($\because$ $\mathbb{Z}$, $\mathbb{Z} \setminus \{0\}$ are countable $\Rightarrow$ $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is countable : $\phi : (m, n) \longrightarrow \frac{m}{n}$ is a surjection between $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{Q}$)

Prop : $S = \{(s_n)_{n=1}^{\infty} : s_n \in \{0, 1\}\}$ is the set of all seq of 0's and 1's : S is not countable.

Proof : • Suppose S is countable : S is ∞ so $\exists$ a bijection $f : \mathbb{N} \longleftrightarrow S$ $\Rightarrow$ Enumerate $S = \{s^{(1)}, s^{(2)}, \dots\}$
• Construct the seq (s_n) : $s_n = \begin{cases} 1, & \text{if } s_n^{(n)} = 0 \\ 0, & \text{if } s_n^{(n)} = 1 \end{cases}$ $\rightsquigarrow (s_n)$ is a seq of 0's & 1's $\Rightarrow (s_n) \in S$.
• But (s_n) differs from any $s^{(n)}$ since the n^{th} element is different $\Rightarrow (s_n) \notin S$ $\longrightarrow \boxed{\Rightarrow \Leftarrow}$

Corr : The interval $[0, 1]$ in $\mathbb{R}$ is uncountable

Proof : • Take $S = \{(s_n)_{n=1}^{\infty} : s_n \in \{0, 1\}\}$: Let $s = (s_n)_{n=1}^{\infty} \in S$ and define $f : S \to [0, 1]$ by $f(s) = \sum_{n=1}^{\infty} 2^{-n} s_n$
• Binary rep of $x \in [0, 1]$ $\Rightarrow$ f is onto BUT f is NOT 1-1 since $(1, 0, 0, \dots) \xrightarrow{f} \frac{1}{2}$, $(0, 1, 1, \dots) \xrightarrow{f} \sum_{n=2}^{\infty} (\frac{1}{2})^n = \frac{1}{2}$ both map to $\frac{1}{2}$.
• Let S_0 be the set of all seq that terminate in 1's : Then S_0 is countable
• Then $S_1 = S \setminus S_0$ is uncountable
• So $f : S \to [0, 1)$, $f(s) = \sum_{n=1}^{\infty} 2^{-n} s_n$ is 1-1 and onto $\Rightarrow [0, 1)$ is not countable as S is not countable

Corr : $a < b$ in $\mathbb{R}$: Then $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ is not countable.

Proof : By previous corr : $\phi : [0, 1] \longleftrightarrow [a, b]$: $\phi(x) = (b - a)x + a$ is a bijection $\Rightarrow [a, b]$ is uncountable $\Rightarrow (a, b)$ is uncountable

Dense sets : X a metric space : a set $S \subset X$ is dense in X if $X = cl(S)$
i.e. $\forall x \in X$: $\exists$ a seq $(x_n) \in S$ ST $\lim_{n \to \infty} x_n = x \in X$

Prop : S is dense in X $\overset{①\,②}{\Longleftrightarrow}$ Any nonempty open set in X contains an element of S.

Proof : ($\Rightarrow$) Suppose ② fails : $\exists$ a nonempty open set O in X ST $O \cap S = \phi$ $\Rightarrow$ $S \subset O^c$ with O^c closed (as O is open)
But $cl(S)$ is the smallest open set containing S $\Rightarrow$ $S \subset cl(S) \subset O^c \neq X$ $\Rightarrow$ $cl(S) \neq X$ $\Rightarrow$ S is not dense in X.
$\hookrightarrow$ ($\because$ $O \neq \phi \Rightarrow O^c \neq \phi^c = X$)

($\Leftarrow$) Suppose ① fails : Then $X \neq cl(S)$
Let $O = (cl(S))^c$: O is nonempty, open and $O \cap cl(S) = \phi$
$\hookrightarrow$ ($\because$ $O^c = cl(S) \neq X \Rightarrow O \neq X^c = \phi$)

(IV) Separability:

separable metric space: A metric space X is separable if it contains a countable dense set.

Ex: $(\mathbb{R}, |\cdot|)$ is separable (::) $\mathbb{Q}$ is dense in $\mathbb{R}$, $\mathbb{Q}$ is countable.

Prop: $(X, \|\cdot\|)$ a normed space: $\dim X < \infty \Rightarrow X$ is separable

⚠ if $(X, \|\cdot\|)$ has $\dim X = \infty$
X could be separable $\to L^p[a,b], p<\infty$
but X may not be separable $\to L^\infty[0,1]$

Corr: $\mathbb{R}^n$ is separable (::) $\dim \mathbb{R}^n = n < \infty$

Proof:
- choose a basis: $\{e_1, e_2, \dots, e_N\}$ of X: let $S = \{r_1 e_1 + \dots + r_N e_N : r_k \in \mathbb{Q}, k = 1, 2, \dots, N\}$
- Let $f: S \to \underbrace{\mathbb{Q} \times \dots \times \mathbb{Q}}_{N \text{ times}} = \mathbb{Q}^N$, ST $f(x) = (r_1, r_2, \dots, r_N)$ where $x = r_1 e_1 + \dots + r_N e_N$
- Let $x \in X$ and $\varepsilon > 0$: write $x = \alpha_1 e_1 + \dots + \alpha_N e_N$ and pick a rational $r_k \in \mathbb{Q}$ ST $|\alpha_k - r_k| < \frac{\varepsilon}{N \cdot \|e_k\|}$ $\forall k \le n$
- Let $y = \sum_{k=1}^{N} r_k e_k \Rightarrow \|x - y\| = \|\sum_{k=1}^{N} (\alpha_k - r_k) e_k\| \le \sum_{k=1}^{N} |\alpha_k - r_k| \cdot \|e_k\| < \sum_{k=1}^{N} \frac{\varepsilon}{N \cdot \|e_k\|} \|e_k\| = \varepsilon. \Rightarrow x = y \Rightarrow x \in cl(S)$
  $\Rightarrow X = cl(S)$
  - so $y \in S \Rightarrow y \in cl(S)$

Prop: Any compact metric space is separable

Proof:
- X is compact $\Rightarrow X$ is totally bounded $\Rightarrow \forall \varepsilon > 0$, X can be covered by finitely many open balls of radius ε.
- Let $N > 0$ be an integer: take $\varepsilon = \frac{1}{N}$, and let $x_1^{(N)}, \dots, x_{n_N}^{(N)}$ be pts in X ST $X = B(x_1^{(N)}, \frac{1}{N}) \cup \dots \cup B(x_{n_N}^{(N)}, \frac{1}{N})$
- Let $S = \bigcup_{N=1}^{\infty} \{x_1^{(N)}, \dots, x_{n_N}^{(N)}\}$: S is countable (::) countable union of countable sets.
- Let $x \in X$: then $\exists k_n$, $1 \le k_n \le n_N$ ST $x \in B(x_{k_N}^{(N)}, \frac{1}{N})$ $\forall N = 1, 2, \dots$ and $x_{k_N}^{(N)} \in S \Rightarrow (x_{k_N}^{(N)}) \in S$.
- $d(x, x_{k_N}^{(N)}) < \frac{1}{N} \Rightarrow (x_{k_N}^{(N)})_{N=1}^{\infty}$ is a seq in X that converges to $x \Rightarrow cl(S) = X$.

Prop: X a metric space: X is separable $\Leftrightarrow \exists$ a countable collect° $\{O_n\}_{n=1}^{\infty}$ of **nonempty** open sets in X ST any open set in X can be written as a union of subcollect° of $\{O_n\}_{n=1}^{\infty}$

Proof: ($\Rightarrow$) $S = \{x_n\}_{n=1}^{\infty}$ a countable dense set in X: (::) X is separable.
- Consider a family of open sets $\{B(x_n, \frac{1}{n}) : n, m \in \mathbb{N}\} \to$ countable collect° since indexed by countable set $(n,m) \in \mathbb{N} \times \mathbb{N}$
- claim: any open set in X can be written as a union of a subcoll° of $\{B(x_n, \frac{1}{n}) : n, m \in \mathbb{N}\}$
  (::) Let O be an open set in X, $x \in O$: then $\exists m > 0$ ST $B(x, \frac{1}{m}) \subset O$: m depends on $x \to m_x$
  S dense in $X \Rightarrow \exists n > 0$ ST $d(x_n, x) < \frac{1}{2m_x}$: n depends on $x \to n_x$. $\Bigg] \Rightarrow$ so $x \in B(x_{n_x}, \frac{1}{2m_x})$
  if $z \in B(x_{n_x}, \frac{1}{2m_x})$: $d(x, z) \le d(x, x_{n_x}) + d(x_{n_x}, z) < \frac{1}{2m_x} + \frac{1}{2m_x} = \frac{1}{m_x} \Rightarrow z \in B(x, \frac{1}{m_x})$
  So $\forall x \in O$: $\exists n_x, m_x \in \mathbb{N}$ ST: $x \in B(x_{n_x}, \frac{1}{2m_x}) \subset B(x, \frac{1}{m_x}) \subset O \Rightarrow O = \bigcup_{x \in O} B(x_{n_x}, \frac{1}{2m_x})$
  But $I = \{(n_x, 2m_x) : x \in O\} \subset \mathbb{N} \times \mathbb{N}$ is countable $\Rightarrow O = \bigcup_{(n,m) \in I} B(x_n, \frac{1}{m})$ is countable.

($\Leftarrow$) $\exists \{O_n\}_{n=1}^{\infty}$ ST any open set in X is a union of a subcoll° of $\{O_n\}_{n=1}^{\infty}$ with O_n open & nonempty.
Pick $x_n \in O_n$, let $S = \{x_n\}_{n=1}^{\infty} \Rightarrow$ Any open set O in X contains some $O_n \Rightarrow O$ contains x_n
So any nonempty open set in X has a nonempty intersection with $S \Rightarrow S$ is dense in $X \Rightarrow X$ is separable.

Prop: X a metric space, $Y \subset X$ a metric subspace ($\to$ same metric!): X is separable $\Rightarrow Y$ is separable

Proof:
- X is separable $\Rightarrow \exists$ a countable collect° $\{O_n\}_{n=1}^{\infty}$ of nonempty open sets in X ST every open set in X can be written as a union of some subcollect° of $\{O_n\}_{n=1}^{\infty}$
- Set $V_n = Y \cap O_n \Rightarrow V_n$ is open in $Y \Rightarrow$ we have a family of open sets $\{V_n\}_{n=1}^{\infty} \to$ if some V_n are empty: drop them and reenumerate the seq.
- Let V in Y be open: then $\exists$ an open set $O \subset X$ ST: $V = Y \cap O$.
- $\exists I \subset \mathbb{N}$ ST $O = \bigcup_{i \in I} O_i \Rightarrow V = Y \cap O = \bigcup_{i \in I} (Y \cap O_i) = \bigcup_{i \in I} V_i \Rightarrow \{V_n\}_{n=1}^{\infty}$ is a countable collect° of nonempty open sets in Y ST any open set in Y can be written as a union of some subcoll° of $\{V_n\}_{n=1}^{\infty} \Rightarrow Y$ is separable.

Prop: X a separable metric space, $\{B(x_i, r_i)\}_{i \in I}$ a family of disjoint open balls in $X \Rightarrow I$ is a countable set

Proof:
- Let $\{y_1, y_2, \dots\}$ be a countable dense set in $X \Rightarrow \forall i \in I$. $\exists n(i) \in \mathbb{N}$ ST $y_{n(i)} \in B(x_i, r_i)$
- if $i \ne i' \Rightarrow y_{n(i)} \ne y_{n(i')}$ because the balls are disjoint, So $n(i) \ne n(i')$
- So the map $f: I \to \mathbb{N}$, $f(i) := n(i)$ is injective $\Rightarrow I$ is in bijection with a subset of $\mathbb{N} \Rightarrow I$ is countable!

$\Rightarrow$ use to show lack of separability: Given X: find an uncountable coll° of disjoint open balls in $X \Rightarrow X$ is NOT separable!

III Density of Polynomials

- $C([0,1])$ = Vspace of all cont funct^{n} $f: [0,1] \to \mathbb{R}$; $\|f\|_\infty = \max_{0\le x\le 1} |f(x)|$
- polynomials: $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

Thm: Let $\varepsilon > 0$ and $f \in C([0,1])$:

$\exists$ a poly $p(x)$ ST $\|f-p\|_\infty = \max_{0\le x\le 1} |f(x)-p(x)| < \varepsilon$

Moreover: can choose p to be a Bernstein poly: $P_n(x) = \sum_{k=0}^{n} f(\frac{k}{n}) \cdot \binom{n}{k} x^k (1-x)^{n-k}$

then $\forall \varepsilon > 0$, $\exists\, n(\varepsilon)$ ST $\|f - P_n\|_\infty < \varepsilon$ $\forall n \ge n(\varepsilon)$

Proof:
- $nx = \sum_{k=0}^{n} k \binom{n}{k} x^k (1-x)^{n-k}$: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \Rightarrow nx(x+y)^{n-1} = x\frac{\partial}{\partial x}(x+y)^n = \sum_{k=0}^{n} k\binom{n}{k} x^k y^{n-k}$ — take $y = 1-x$
- $n(n-1)x^2 + nx = \sum_{k=0}^{n} k^2 \binom{n}{k} x^k (1-x)^{n-k}$: $n(n-1)x^2(x+y)^{n-2} = x^2 \frac{\partial^2}{\partial x^2}(x+y)^n = \sum_{k=0}^{n} k(k-1) x^k y^{n-k} \xRightarrow{y=1-x} n(n-1)x^2 = \sum_{k=0}^{n} k(k-1) x^k (1-x)^{n-k}$

  add $nx = \sum_{k=0}^{n} k \binom{n}{k} x^k (1-x)^{n-k}$
- Let $\boxed{r_k(x) := \binom{n}{k} x^k (1-x)^{n-k}}$: $\boxed{\sum_{k=0}^{n} r_k(x) = 1 \ ; \ \sum_{k=0}^{n} k\, r_k(x) = nx \ ; \ \sum_{k=0}^{n} k^2 r_k(x) = n(n-1)x^2 + nx}$
- $\boxed{\sum_{k=0}^{n} (nx-k)^2 r_k(x) = nx(1-x)}$ $(\because)$ $\sum_{k=0}^{n} (nx-k)^2 r_k(x) = n^2x^2 \sum_{k=0}^{n} r_k(x) - 2n \sum_{k=0}^{n} k\, r_k(x) + \sum_{k=0}^{n} k^2 r_k(x) = n^2x^2 - 2n^2x^2 + n(n-1)x^2 + nx$

  $= -nx^2 + nx = nx(1-x)$.
- Let $f \in C([0,1])$: then f is unif cont on $[0,1]$

  So $\forall \varepsilon > 0$, $\exists \delta > 0$ ST: if $x, y \in [0,1]$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}$.
- Let $M = \|f\|_\infty = \max_{0\le x\le 1} |f(x)|$, and let $P_n(x) := \sum_{k=0}^{n} f(\frac{k}{n}) r_k(x)$ with $r_k(x) := \binom{n}{k} x^k (1-x)^{n-k}$

  So $|f(x) - P_n(x)| = |f(x) \sum_{k=0}^{n} r_k(x) - \sum_{k=0}^{n} f(\frac{k}{n}) r_k(x)| = |\sum_{k=0}^{n} (f(x) - f(\frac{k}{n})) r_k(x)|$

  $\le \sum_{k,\ |x-\frac{k}{n}|<\delta} \underbrace{|f(x) - f(\tfrac{k}{n})|}_{<\varepsilon/2} r_k(x) + \sum_{k,\ |x-\frac{k}{n}|\ge\delta} \underbrace{|f(x) - f(\tfrac{k}{n})|}_{<2M} r_k(x) \le \frac{\varepsilon}{2} \underbrace{\sum_{k,\ |x-\frac{k}{n}|<\delta} r_k(x)}_{\le \sum_{k=0}^{n} r_k(x) = 1} + 2M \sum_{k,\ |x-\frac{k}{n}|\ge\delta} r_k(x)$

  $\le \frac{\varepsilon}{2} + 2M \sum_{k,\ |x-\frac{k}{n}|\ge\delta} \frac{(x-\frac{k}{n})^2}{\delta^2} r_k(x) \le \frac{\varepsilon}{2} + \frac{2M}{\delta^2} \sum_{|x-\frac{k}{n}|\ge\delta} \frac{(nx-k)^2}{n^2} r_k(x) \le \frac{\varepsilon}{2} + \frac{2M}{n^2\delta^2} \sum_{k=0}^{n} (nx-k)^2 r_k(x) = \frac{\varepsilon}{2} + \frac{2M}{n^2\delta^2} n\underbrace{x(1-x)}_{\le \frac{1}{4} \text{ on } [0,1]}$

  So $|f(x) - P_n(x)| < \frac{\varepsilon}{2} + \frac{M}{2n\delta^2}$ $\forall x \in [0,1]$, $\forall n \in \mathbb{N}$
- Let $n(\varepsilon)$ be the smallest integer ST $n(\varepsilon) \ge \frac{M}{\varepsilon\delta^2} \Rightarrow \forall n \ge n(\varepsilon)$: $\frac{M}{2n\delta^2} \le \frac{\varepsilon}{2}$

  So $\forall n \ge n(\varepsilon)$, $\forall x \in [0,1]$: $|f(x) - P_n(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\Rightarrow$ given $f \in C([0,1])$, $\varepsilon > 0$: Any Bernstein poly with $n \ge n(\varepsilon)$ satisfies $\|f - P_n\|_\infty < \varepsilon$. ☑

Corr: In addition, if f is Lipschitz on $[0,1]$: $n(\varepsilon) = \frac{4MK^2}{\varepsilon^3} \to \forall \varepsilon > 0$, if $n \ge n(\varepsilon)$ then $\|f - P_n\|_\infty < \varepsilon$

Proof: f is Lipschitz $\Rightarrow \exists K > 0$ ST $|f(x) - f(y)| < K|x-y|$ $\forall x, y \in [0,1]$

if f cont & diff on $(0,1)$ and $\tilde{M} = \sup_{x\in(0,1)} |f'(x)| < \infty \Rightarrow$ MVT: $\exists c \in [0,1]$ ST $\left|\frac{f(x) - f(y)}{x-y}\right| = f'(c) < \tilde{M} < \infty$. $\forall x, y \in X$.

$\Rightarrow$ Take $K = \tilde{M}$.

So $\forall \varepsilon > 0$: take $\delta = \frac{\varepsilon}{2M}$: if $|x-y| < \delta$, then $|f(x) - f(y)| < K|x-y| \le K\delta = \frac{\varepsilon}{2}$

So $n(\varepsilon) \ge \frac{M}{\varepsilon\delta^2} = \frac{M}{\varepsilon(\frac{\varepsilon}{2K})^2} = \frac{4K^2M}{\varepsilon^3}$

IV Separability of $C([a,b])$:

Let $\mathcal{P}_N = \{ r_0 + r_1 x + \dots + r_N x^N : r_k \in \mathbb{Q}, 1 \le k \le n \}$ → set of all poly of degree $\le N$ with rational coeffs

Prop: $S = \bigcup_{N=0}^{\infty} \mathcal{P}_N$ is countable

Proof: Let $f : \mathcal{P}_n \longrightarrow \underbrace{\mathbb{Q} \times \dots \times \mathbb{Q}}_{N+1 \text{ times}} = \mathbb{Q}^{N+1}$ st $f(P) = (r_0, r_1, \dots, r_N)$ for $P(x) = r_0 + r_1 x + \dots + r_N x^N$

f is clearly a bijection (∵ poly uniquely determined by coeffs) ⟹ $\mathcal{P}_N$ is countable ⟹ S is countable (countable union of countable set)

Prop: $(C([0,1]), \|\cdot\|_\infty)$ is separable

Proof: claim: S is dense in $(C([0,1]), \|\cdot\|_\infty)$

Let $f \in C([0,1])$, and $\varepsilon > 0$:

- choose poly $p(x) = a_0 + a_1 x + \dots + a_N x^N$ st $\|f - p\|_\infty < \frac{\varepsilon}{2}$
- Let $r_0, r_1, \dots, r_N \in \mathbb{Q}$ st $|a_k - r_k| < \frac{\varepsilon}{2(n+1)}$ for $0 \le k \le n$ → set $g(x) = r_0 + r_1 x + \dots + r_N x^N$

So $|p(x) - g(x)| = \left| \sum_{k=0}^{N} x^k (a_k - r_k) \right| \le \sum_{k=0}^{N} \underbrace{x^k}_{\le 1 \ 0 \le x \in [0,1]} |a_k - r_k| \le \sum_{k=0}^{n} 1 \cdot \frac{\varepsilon}{2(n+1)} = \frac{\varepsilon}{2} \cdot \frac{n+1}{n+1} = \frac{\varepsilon}{2} \Rightarrow \|p - g\|_\infty = \sup_{x \in [0,1]} |p(x) - g(x)| < \frac{\varepsilon}{2}$

So $g \in S$ and $\|f - g\|_\infty \le \|f - p\|_\infty + \|p - g\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ ⟹ S is dense in $(C([0,1]), \|\cdot\|_\infty)$

Since S is countable & dense ⟹ $(C([0,1]), \|\cdot\|_\infty)$ is separable!

Prop: $-\infty < a < b < \infty$: polynomials are dense in $(C([a,b]), \|\cdot\|_\infty)$

Corr: $(C([a,b]), \|\cdot\|_\infty)$ is separable

Proof:
- define: $F : C([0,1]) \longrightarrow C([a,b])$ by $(F(f))(x) \longmapsto f\left(\frac{x-a}{b-a}\right)$ $\forall x \in [a,b]$
  - → F is linear ($F(\alpha f + \beta g) = \alpha F(f) + \beta F(g)$), F is bijective and $\|F(f)\|_\infty = \|f\|_\infty$ (as) $\|F(f)\|_\infty = \sup_{x \in [a,b]} |(F(f))(x)| = \sup_{x \in [a,b]} \left| f\left(\frac{x-a}{b-a}\right) \right| = \sup_{t \in [0,1]} |f(t)| = \|f\|_\infty$ $\left(t = \frac{x-a}{b-a}\right)$
  - → $F^{-1} : C([a,b]) \longrightarrow C([0,1])$: $(F^{-1}(g))(x) = g(a + (b-a)x)$ $\forall x \in [0,1]$
  - → F and F^{-1} map polys to polys: linear change of var
- Let $g \in C([a,b])$ and $\varepsilon > 0$: then $F^{-1}(g) \in C([0,1])$.
- S is countable: ∃ a poly st $\|F^{-1}(g) - p\|_\infty < \varepsilon \Rightarrow \|F(F^{-1}(g)) - F(p)\|_\infty < \varepsilon \Rightarrow \|g - \overset{\text{a poly}}{F(p)}\|_\infty < \varepsilon$: S countable in $C([0,1])$ ⟹ S countable in $C([a,b])$
- S is dense in $(C([a,b]), \|\cdot\|_\infty)$: same arg: S dense in $C([0,1])$ ⟹ S dense in $C([a,b])$.

⟹ S is countable & dense in $(C([0,1]), \|\cdot\|_\infty)$ ⟹ $F(S)$ is countable & dense in $(C([a,b]), \|\cdot\|_\infty)$

So $(C([0,1]), \|\cdot\|_\infty)$ is separable ⟹ $(C([a,b]), \|\cdot\|_\infty)$ is separable

Prop: $(C([a,b]), \|\cdot\|_p)$ is separable if $1 \le p \le \infty$, with $\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$

Proof:
- $p = \infty$ → previous prop
- $1 \le p < \infty$: $\forall f, g \in C([a,b])$: $\|f - g\|_p = \left(\int_a^b |f(x) - g(x)|^p dx \right)^{1/p} \le \sup_{x \in [a,b]} |f(x) - g(x)| \cdot (b-a)^{1/p} \le \|f - g\|_\infty \cdot (b-a)^{1/p}$

S is countable: Given $\varepsilon > 0$: choose a poly $q(x)$ st $\|f - q\|_\infty < \frac{\varepsilon}{(b-a)^{\frac{1}{p}}} \Rightarrow \|f - q\|_p \le \|f - q\|_\infty \cdot (b-a)^{1/p} < \frac{\varepsilon}{(b-a)^{1/p}} \cdot (b-a)^{1/p} = \varepsilon$

Similar estimates: if S is dense in $(C[a,b], \|\cdot\|_\infty)$ then S is dense in $(C[a,b], \|\cdot\|_p)$, $1 \le p < \infty$

(V) Separability & Product spaces:

Prop: (1) $(X_1,d_1), \dots, (X_n,d_n)$: finitely many separable metric spaces.
Then their Cartesian product (X,d) is separable.

(Midterm 2009) (2) (X_k,d_k), $k=1,2,\dots$ a seq of separable metric spaces:
Then their Cartesian product (X,d) is separable.

Proof: (2) Let $\overline{x}_k \in X_k$ be fixed points:

- X_k is separable $\Rightarrow \exists S_k \subset X_k$ with S_k countable & dense in X_k.
- $\forall N \in \mathbb{N}$: $\widetilde{S}_N := \left\{ x=(x_k)_{k=1}^{\infty} : \begin{cases} x_k \in S_k, & 1 \le k \le N \\ x_k = \overline{x}_k, & N<k \end{cases} \right\}$

So a seq: $(x_k)_{k=1}^{\infty} \in \widetilde{S}_N \iff$ it is of the form: $(s_1, s_2, \dots, s_N, \overbrace{\overline{x}_{N+1}, \overline{x}_{N+2}, \dots}^{\text{fixed!}})$ where $s_k \in S_k$ for $1 \le k \le N$.

So $\widetilde{S}_N$ is in bijective corresp with $S_1 \times S_2 \times \dots \times S_N$ as the tail is fixed $\Rightarrow \widetilde{S}_N$ is countable

So $S := \bigcup_{N=1}^{\infty} \widetilde{S}_N$ is countable (($\because$) countable union of countable sets)

- claim: S is dense in (X,d):

Let $x=(x_k)_{k=1}^{\infty}$ be a given seq and let $\varepsilon>0$ given:
choose N ST $\boxed{\sum_{k=N+1}^{\infty} 2^{-k} < \frac{\varepsilon}{2}}$; and $s_k \in S_k$ ST $\boxed{d(x_k,s_k) < \frac{\varepsilon}{2N}}$; and let $y=(s_1,s_2,\dots,s_N,\overline{x}_{N+1},\overline{x}_{N+2},\dots) \Rightarrow y \in S$

Also, $d(x,y) = \sum_{k=1}^{N} \underbrace{2^{-k}}_{\le 1} d_k(x_k,s_k) \cdot \underbrace{\frac{1}{1+d_k(x_k,s_k)}}_{\le 1} + \sum_{k=N+1}^{\infty} 2^{-k} \underbrace{\frac{d(x_k,\overline{x}_k)}{1+d_k(x_k,\overline{x}_k)}}_{\le 1} \le \sum_{k=1}^{N} d_k(x_k,s_k) + \sum_{k=N+1}^{\infty} 2^{-k} < \frac{\varepsilon}{2N} N + \frac{\varepsilon}{2} = \varepsilon$

Lemma: X a set, $f: X \to [0,\infty)$ a bounded func^n, $G: [0,\infty) \to [0,\infty)$ continuous & increasing f^n
then $G\left(\sup_{x\in X} f(x)\right) = \sup_{x \in X} G(f(x))$

Proof:
- Let $M = \sup_{x\in X} f(x)$ be the least UB of $f(X) = \{f(x) : x \in X\} \Rightarrow \begin{cases} M \ge f(x) \ \forall x \in X \\ \forall \varepsilon>0,\ M-\varepsilon \text{ is not a UB} \Rightarrow \forall \varepsilon>0, \exists x_\varepsilon \in X \text{ ST } M-\varepsilon < f(x_\varepsilon) \end{cases}$
- $G(M) \ge \sup_{x\in X} G(f(x))$: $M \ge f(x)\ \forall x \in X$ and G increasing $\Rightarrow G(M) \ge G(f(x))\ \forall x \in X \Rightarrow G(M)$ is a UB for $\{G(f(x))\}_{x\in X}$ $\Rightarrow G(M) \ge \sup_{x\in X} G(f(x))$
- $G(M) \le \sup_{x\in X} G(f(x))$: Let $\varepsilon>0$ ST $M-\varepsilon>0$: so $\exists x_\varepsilon \in X$ ST $M-\varepsilon < f(x_\varepsilon) \Rightarrow G(M-\varepsilon) \le G(f(x_\varepsilon)) \le \sup_{x\in X} G(f(x))$
So $\forall \varepsilon>0$, $G(M-\varepsilon) \le \sup_{x\in X} G(f(x))$: let $\varepsilon \to 0$, then G cont at M $\Rightarrow G(M) \le \sup_{x\in X} G(f(x))$

Prop: (Midterm 2009) Let $C_b(\mathbb{R})$ be the Vspace of all bounded cont f^n $f: \mathbb{R} \to \mathbb{R}$, with $\|f\| = \sup_{x\in\mathbb{R}} |f(x)| < \infty$.
(1) $\|\cdot\|$ is a norm on $C_b(\mathbb{R})$ & $(C_b(\mathbb{R}), \|\cdot\|)$ is a Banach space.
(2) $C_b(\mathbb{R})$ is NOT separable!

Proof: (1) Exercise

(2) Let $X = C_b(\mathbb{R}) \times C_b(\mathbb{R})$ and $d(F,G) = \sqrt{\left(\sup_{x\in\mathbb{R}} |f_1(x)-g_1(x)|\right)^2 + \left(\sup_{x\in\mathbb{R}} |f_2(x)-g_2(x)|\right)^2}$ where $F,G \in X$, $F=(f_1,f_2)$, $G=(g_1,g_2)$ and $f_1,f_2,g_1,g_2 \in C_b(\mathbb{R})$.
Clearly if (X,d) is not separable then $C_b(\mathbb{R})$ is not separable.

- Let $F_t = (\cos(tx), \sin(tx))$, $F_s = (\cos(sx), \sin(sx))$: t,s are parameters, x is the variable.

if $t \ne s$: $d(F_t,F_s) = \sqrt{\left(\sup_{x\in\mathbb{R}} |\cos(tx)-\cos(sx)|\right)^2 + \left(\sup_{x\in\mathbb{R}} |\sin(tx)-\sin(sx)|\right)^2} \overset{\text{Lemma}}{=} \sqrt{\sup_{x\in\mathbb{R}} \left((\cos(tx)-\cos(sx))^2\right) + \sup_{x\in\mathbb{R}} \left((\sin(tx)-\sin(sx))^2\right)}$

$\ge \sqrt{\sup_{x\in\mathbb{R}} \left(|\cos(tx)-\cos(sx)|^2 + |\sin(tx)-\sin(sx)|^2 \right)} \overset{\text{Lemma}}{=} \sup_{x\in\mathbb{R}} \sqrt{(\cos(tx)-\cos(sx))^2 + (\sin(tx)-\sin(sx))^2}$

So $d(F_t,F_s) \ge \sup_{x\in\mathbb{R}} \sqrt{2-2(\cos(tx)\cos(sx)+\sin(tx)\sin(sx))} = \sup_{x\in\mathbb{R}} \sqrt{2-2\cos((t-s)x)} \ge \sqrt{2-2\cos\left((t-s)\cdot\frac{\pi/2}{(t-s)}\right)} = \sqrt{2} > 1$

So $d(F_t,F_s) > 1 \quad \forall t \ne s$

So $\{B(F_t,\frac{1}{2})\}_{t\in\mathbb{R}}$ is a disjoint uncountable family of open balls in $(X,d) \Rightarrow X$ is not separable.

Ascoli - Arzela theorem:

Prop: X a compact metric space

$C(X)$: the vspace of all cont funct$^{\circ}$ $f: X \to \mathbb{R}$: $\|f\| = \sup_{x \in X} |f(x)|$ is a norm on $C(X)$ ($(\cdot\cdot)$ X is compact, so: f achieves its min/max value & f is uniformly continuous.)

then $(C(X), \|\cdot\|)$ is a Banach space.

Proof: We proved this when $X = [a,b]$ is an interval $\to$ the general proof is identical.

Recall: • $(Y, \|\cdot\|)$ is a fin. dim vspace: $S \subset Y$ is compact $(S, \|\cdot\|)$ $\Leftrightarrow$ S is closed & bounded in $(Y, \|\cdot\|)$

• $\dim C(X) = \infty$ in general: $D \subset C(X)$ is compact $\overset{A-A}{\Longleftrightarrow}$ D is closed, bounded AND equicontinuous in $(C(X), \|\cdot\|)$

Equicontinuity: A collection $D \subset C(X)$ of real-valued functions on a metric space X is equicontinuous if:

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall f \in D)(\forall x, y \in X)\left(d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon\right)$$

$\hookrightarrow$ independent of f

$\neq$ unif cont: $(\forall \varepsilon > 0)(\forall f \in D)(\exists \delta_f > 0)(\forall x, y \in X)(d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon)$

Ascoli-Arzela theorem: X a compact metric space, $D \subset C(X)$:

$$D \text{ is compact in } (C(X), \|\cdot\|) \overset{①\quad②}{\Longleftrightarrow} D \text{ is closed, bounded AND Equicontinuous}$$

Proof: **(1) ⇒ (2)**: • D is a compact metric subspace of $(C(X), \|\cdot\|)$ $\Rightarrow$ D is closed & bounded

• Goal: D is equicontinuous

Fix $\varepsilon > 0$ and consider $\{B(f, \frac{\varepsilon}{3})\}_{f \in D}$: This is an open cover of D since we use all $f \in D$.

D is compact $\Rightarrow$ $\exists$ a finite subcover: $\exists f_1, f_2, \dots, f_\ell \in D$ ST $D \subset B(f_1, \frac{\varepsilon}{3}) \cup \dots \cup B(f_\ell, \frac{\varepsilon}{3})$ (with $\ell < \infty$)

Each $f_k \in D$ is unif cont on X $\Rightarrow$ $\exists \delta_k > 0$ ST $(d(x,y) < \delta_k \Rightarrow |f_k(x) - f_k(y)| < \frac{\varepsilon}{3})$

Set $\delta = \min_{1 \le \delta \le \ell} \{\delta_k\} > 0$ and let $f \in D$: then $f \in B(f_k, \frac{\varepsilon}{3})$ for some $1 \le k \le \ell$ $\Rightarrow$ $|f - f_k| < \frac{\varepsilon}{3}$ ($(\cdot\cdot)$ $D \subset \bigcup_{k=1}^{\ell} B(f_k, \frac{\varepsilon}{3})$)

So if $d(x,y) < \delta$: $|f(x) - f(y)| \le |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$ ☑

(2) ⇒ (1): Summary $\to$ see next page for full proof

Step 1: X is compact $\Rightarrow$ X is separable $\Rightarrow$ pick a countable dense set $S = \{x_j\}_{j=1}^{\infty} \subset X$.

Step 2: D is bounded $\Rightarrow$ $\exists M > 0$ ST $\sup_{x \in X} |f(x)| = \|f\| \le M$: let $(f_n)_{n=1}^{\infty}$ be a seq in $C(X)$

Step 3: $\prod_{j=1}^{\infty} [-M, M]$ is compact $\Rightarrow$ find a subseq $(f_{n_k})_{k=1}^{\infty}$ ST: $\lim_{k \to \infty} f_{n_k}(x_j) = g(x_j)$ exists $\forall j$.

Step 4: Equicontinuity $\Rightarrow$ g is uniformly continuous on S.

Extension by unif. cont (*) $\Rightarrow$ g extends to a unif cont function $g: X \to \mathbb{R}$.

Step 5: (g is unif cont) + (D is equicontinuous) + (S is dense in X) $\Longrightarrow$ $\lim_{k \to \infty} f_{n_k}(x) = g(x)$ $\forall x \in X$.

Step 6: X is compact $\Rightarrow$ $\lim_{k \to \infty} \|f_{n_k} - g\| = 0$

Recall: Extension by unif. cont: X, Y metric spaces, Y complete: if S dense in X ($X = cl(S)$) and $f: S \to Y$ is unif cont

$\Rightarrow$ f has a unique extension to a unif cont function: $f: X \to Y$.

Note: Step 1-5 work for any separable metric space X : compactness is only used for Step 6!

Thm: X a separable metric space, $f_n: X \to \mathbb{R}$ a seq of continuous functions.

if: (1) the seq $(f_n)_{n=1}^{\infty}$ is bounded ie: $(\exists M > 0)(\forall n \in \mathbb{N})(\sup_{x \in X} |f_n(x)| \le M)$

And: (2) the seq $(f_n)_{n=1}^{\infty}$ is equicontinuous ie: $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall n \in \mathbb{N})(d(x,y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon)$

$\Rightarrow$ $\exists$ a subseq $(f_{n_k})_{k=1}^{\infty}$ ST: $g(x) := \lim_{k \to \infty} f_{n_k}(x)$ exists $\forall x \in X$

Moreover, g will be uniformly continuous on X.

Proof: (Arzela–Ascoli theorem) X a compact metric space, $D \subset C(X)$:

D is closed, bounded & equicontinuous $\Longrightarrow$ $(D, \|\cdot\|)$ is compact in $(C(X), \|\cdot\|)$

- **Step 1:** X is compact $\Rightarrow$ X is separable $\Rightarrow$ Let $S = \{x_j\}_{j=1}^{\infty}$ be a countable dense set in X.

- **Step 2:**
  - D is bounded $\Rightarrow \exists M > 0$ ST: $\sup_{x \in X} |f(x)| = \|f\| \leq M \quad \forall f \in D$
  - Let $(f_n)_{n=1}^{\infty}$ be a seq in D: GOAL: $\exists$ a subseq that converges in $(C(X), \|\cdot\|)$

- **Step 3:**
  - Consider the seq: $(f_n(x_j))_{j=1}^{\infty}$ for $n = 1, 2, 3, \dots$
  - f_n are bounded $\Rightarrow (f_n(x_j))_{j=1}^{\infty} \in \prod_{j=1}^{\infty} [-M, M]$ with $d(z, z') = \sum_{j=1}^{\infty} 2^{-j} \frac{|z_j - z'_j|}{1 + |z_j - z'_j|} \quad \forall z, z' \in \prod_{j=1}^{\infty} [-M, M]$
  - $\prod_{j=1}^{\infty} [-M, M]$ is a countable prod of compact metric spaces $\Rightarrow \prod_{j=1}^{\infty} [-M, M]$ is compact $\Rightarrow (f_n(x_j))_{j=1}^{\infty}, n = 1, 2, \dots$ has a convergent subseq in $\prod_{j=1}^{\infty} [-M, M]$

    $\rightarrow$ denote that convergent subseq: $(f_{n_k}(x_j))_{j=1}^{\infty}, k = 1, 2, \dots$: Convergence in the prod space is equivalent to componentwise conv

    So $g(x_j) := \lim_{k \to \infty} f_{n_k}(x_j)$ exists $\forall j = 1, 2, 3, \dots \Rightarrow$ This defines g on $S = \{x_j\}_{j=1}^{\infty}$

- **Step 4:**
  - g is unif cont on S:

    (*) Fix $\varepsilon > 0$: Equicontinuity of $D \Rightarrow \exists \delta > 0$ ST: $d(x_i, x_j) < \delta \Rightarrow |f_{n_k}(x_i) - f_{n_k}(x_j)|$ $\forall x_i, x_j \in S, \forall k \in \mathbb{N}$.

    Take $k \to \infty$: $d(x_i, x_j) < \delta \Rightarrow |g(x_i) - g(x_j)| < \varepsilon \Rightarrow g: S \to \mathbb{R}$ is unif cont on S
  - $\mathbb{R}$ is complete & S is dense in X $\Rightarrow$ Extension by unif cont: g extends uniquely to a unif cont functⁿ: $g: X \to \mathbb{R}$.

- **Step 5:** We know that $\lim_{k \to \infty} f_{n_k}(x) = g(x) \ \forall x \in S$: Goal true $\forall x \in X$!
  - fix $x \in X$ and let $\varepsilon > 0$:

    g unif cont on $X \Rightarrow \exists \delta_1 > 0$ ST: $\forall z', z'' \in X$: $d(z', z'') < \delta \Rightarrow |g(z') - g(z'')| < \frac{\varepsilon}{3}$ } Let $\delta := \min\{\delta_1, \delta_2\}$

    Equicontinuity of $D \Rightarrow \exists \delta_2 > 0$ ST: $\forall k \in \mathbb{N}$: $d(z', z'') < \delta \Rightarrow |f_{n_k}(z') - f_{n_k}(z'')| < \frac{\varepsilon}{3}$

    S dense in $X \Rightarrow \exists j \in \mathbb{N}$ ST: $d(x, x_j) < \delta$ so that:

  $\Rightarrow |g(x) - f_{n_k}(x)| \leq |g(x) - g(x_j)| + |g(x_j) - f_{n_k}(x_j)| + |f_{n_k}(x_j) - f_{n_k}(x)| < \frac{\varepsilon}{3} + |g(x_j) - f_{n_k}(x_j)| + \frac{\varepsilon}{3} \quad \forall k \in \mathbb{N}$ (by choice of δ)

  But $\lim_{k \to \infty} f_{n_k}(x_j) = g(x_j) \Rightarrow \exists K > 0$ ST $\forall k \geq K$: $|f_{n_k}(x_j) - g(x_j)| < \frac{\varepsilon}{3}$.

  So $\forall k \geq K$: $|g(x) - f_{n_k}(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$ $\longrightarrow$ we proved convergence: $\lim_{k \to \infty} f_{n_k}(x) = g(x)$

  But NOT unif conv yet...

  ⚠ x_j depends on the $x \in X$ we fixed $\Longrightarrow$ K depends on x.

- **Step 6:** Goal: $f_{n_k} \to f$ in $(C(X), \|\cdot\|)$ uniformly
  - Let $\varepsilon > 0$: let δ be like in step 5: $d(z', z'') < \delta \Rightarrow \begin{cases} |g(z') - g(z'')| < \varepsilon/3 \\ |f_{n_k}(z') - f_{n_k}(z'')| < \varepsilon/3 \quad \forall k \in \mathbb{N} \end{cases}$
  - Consider $\{B(x_j, \delta)\}_{j=1}^{\infty}$, where $S = \{x_j\}_{j=1}^{\infty}$: $X \subset \bigcup_{j=1}^{\infty} B(x_j, \delta)$: These balls cover X.

    if $x \in X \Rightarrow B(x, \delta) \cap S \neq \emptyset$ (*/ S is dense in X) $\Rightarrow \exists j \in \mathbb{N}$ ST $x_j \in B(x, \delta) \Rightarrow x \in B(x_j, \delta) \Rightarrow$
  - X is compact $\Rightarrow \exists$ a finite subcover of X: $X = B(x_{i_1}, \delta) \cup \dots \cup B(x_{i_\ell}, \delta) \Rightarrow \lim_{k \to \infty} f_{n_k}(x_{i_r}) = g(x_{i_r}) \quad \forall 1 \leq r \leq \ell$.

    since $\ell < \infty \Rightarrow \exists K > 0$ ST $|f_{n_k}(x_{i_r}) - g(x_{i_r})| < \frac{\varepsilon}{3} \quad \forall k \geq K, r = 1, 2, \dots, \ell$ (**) Take $K = \max_{1 \leq r \leq \ell}\{K_r\}$)

  Let $x \in X$: then $x \in B(x_{i_r}, \delta)$ for some $1 \leq r \leq \ell$:

  $\forall k \in \mathbb{N}$: $|g(x) - f_{n_k}(x)| \leq |g(x) - g(x_{i_r})| + |g(x_{i_r}) - f_{n_k}(x_{i_r})| + |f_{n_k}(x_{i_r}) - f_{n_k}(x)| < \underbrace{\frac{\varepsilon}{3}}_{\text{choice of } \delta \sim \text{step 5}} + |g(x_{i_r}) - f_{n_k}(x_{i_r})| + \underbrace{\frac{\varepsilon}{3}}_{\text{choice of } \delta \sim \text{step 5}}$

  So $\forall k \geq K$ (K indep on choice of x_{i_r}): $|g(x_{i_r}) - f_{n_k}(x_{i_r})| < \frac{\varepsilon}{3}$.

  $\Rightarrow \forall k \geq K$: $|g(x) - f_{n_k}(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$ where K indep of x.

  So $\forall k \geq K$: $\sup_{x \in X} |f_{n_k}(x) - g(x)| < \varepsilon \Rightarrow \lim_{k \to \infty} \|f_{n_k} - g\| = \lim_{k \to \infty} \sup_{x \in X} |f_{n_k}(x) - g(x)| = 0$

  So $f_{n_k} \to f$ uniformly in $(C(X), \|\cdot\|)$

  $\hookrightarrow$ K indep of the choice of $x \in X$

Corr: (Reformulation of the A-A theorem) $\longrightarrow$ Useful in applications!

X a compact metric space, $D \subset C(X)$

$cl(D)$ is compact in $(C(X), \|\cdot\|) \iff D$ is bounded and equicontinuous

Proof: $(1) \Rightarrow (2)$: $cl(D)$ is compact $\Rightarrow$ AA: $cl(D)$ is bounded & equicontinuous $\Rightarrow D$ is bounded & equicont (since $D \subset cl(D)$)

$(2) \Rightarrow (1)$: Goal: $cl(D)$ is compact

We know: D is bounded and equicontinuous.

Need: $cl(D)$ is closed, bounded & equicontinuous (so AA $\Rightarrow cl(D)$ is compact)

- $cl(D)$ is closed: obvious
- $cl(D)$ is Bounded: Let $M > 0$ be ST: $\|f\| < M \ \forall f \in D$. Let $g \in cl(D)$ and $(f_n)_{n=1}^{\infty}$ a seq in D ST $\lim_{n\to\infty} \|f_n - g\| = 0$ $\Big\}$ $\|g\| \leq \|g - f_n\| + \|f_n\| \leq \|g - f_n\| + M \ \forall n \in \mathbb{N}$; take $n \to \infty$: $\|g\| \leq M$
- $cl(D)$ is Equicontinuous:

Goal: given $\varepsilon > 0$, $\exists \delta > 0$ ST: $\forall g \in cl(D)$, $(d(x,y) < \delta \Rightarrow |g(x) - g(y)| < \delta)$

We know: D is equicont $\Rightarrow$ fix $\varepsilon > 0$ and choose $\delta > 0$ ST: $\forall f \in D$, $(d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{3})$

So: let $g \in cl(D)$ and let $f \in D$ be ST: $\|g - f\| = \sup_{x\in X} |g(x) - f(x)| < \frac{\varepsilon}{3}$

So if $d(x,y) < \delta$: $|g(x) - g(y)| \leq |g(x) - f(x)| + |f(x) - f(y)| + |f(y) - g(y)| \leq \|g - f\| + |f(x) - f(y)| + \|f - g\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.

So $d(x,y) < \delta \Rightarrow |g(x) - g(y)| < \varepsilon$.

Example 1 = Let $X = [0,1]$ and $k_1, k_2 > 0$ constants.

Let $D \subset C([0,1])$ be a collect° of f° ST:
$$\begin{cases} (1) \ \sup_{x\in X} |f(x)| \leq k_1, \ \forall f \in D \\ (2) \ \text{Each } f \in D \text{ is diff on } (0,1) \text{ and } \sup_{x\in X} |f'(x)| \leq k_2 \ \forall f \in D \end{cases}$$

$\Longrightarrow cl(D)$ is compact in $(C([0,1]), \|\cdot\|)$

Proof: Goal: D is bounded & equicont (So by A-A corr: $cl(D)$ is compact)

- (1) $\Rightarrow D$ is bounded
- MVT $\Rightarrow \forall x, y \in [0,1]$, $\exists c \in (0,1)$ ST $f(x) - f(y) = f'(c) \cdot (x - y)$
- (2) $\Rightarrow |f(x) - f(y)| = |f'(c)| \cdot |x - y| \leq k_2 \cdot |x - y|$
- Fix $\varepsilon > 0$: take $\delta = \frac{\varepsilon}{k_2}$: if $|x - y| < \delta \Rightarrow |f(x) - f(y)| \leq k_2 \cdot |x - y| < k_2 \cdot \delta = k_2 \cdot \frac{\varepsilon}{k_2} = \varepsilon$ ☑

Note: We Need $\sup_{x\in X} |f'(x)| < \infty$: $f(x) = \begin{cases} x \sin(\frac{1}{x}), & 0 < x \leq 1 \\ 0, & x = 0 \end{cases} \Rightarrow f'(x) = \sin(\frac{1}{x}) - \frac{1}{x}\cos(\frac{1}{x}), \ 0 < x < 1$

Take $x_n = \frac{1}{2\pi n}$: $x_n \to 0$ but $f'(x_n) = \sin(2\pi n) - 2\pi n \cos(2\pi n) = -2\pi n \longrightarrow -\infty$

Example 2 = $X = [0,1]$, $k_1, k_2 > 0$ constants.

Let $D \subset C([0,1])$ be a collect° of funct° ST:
$$\begin{cases} (1) \ |f(0)| \leq k_1, \ \forall f \in D \\ (2) \ f \text{ is continuously diff on } [0,1] \ \forall f \in D \\ (3) \ \int_0^1 |f'(x)|^2 dx \leq k_2 \end{cases}$$

$\longrightarrow$ f has a derivative on $(0,1)$, f' is continuous on $[0,1]$ (by extension)

$\Longrightarrow cl(D)$ is compact in $(C([0,1]), \|\cdot\|)$

Proof: Goal: D is bounded & equicontinuous:

- D is bounded: $f(x) = f(x) - f(0) + f(0) = \int_0^x f'(x)\,dx + f(0)$

So $|f(x)| \leq \int_0^x 1 \cdot |f'(x)|\,dx + |f(0)| \underset{\text{Hölder}}{\leq} (\int_0^x 1^2 dt)^{\frac{1}{2}} \cdot (\int_0^x |f'(x)|^2 dx)^{\frac{1}{2}} + |f(0)| \leq \sqrt{x} \cdot \sqrt{k_2} + k_1 \leq 1 \cdot \sqrt{k_2} + k_1 \ (0 \leq x \leq 1)$

So $|f(x)| \leq \sqrt{k_2} + k_1 \ \forall x \in [0,1] \Rightarrow \sup_{x\in X} |f(x)| \leq k_1 + \sqrt{k_2} \Rightarrow \|f\| \leq k_1 + \sqrt{k_2}$: D is bounded

- D is equicontinuous:

Suppose $x < y$: $f(x) - f(y) = \int_x^y f'(t)\,dt \Rightarrow |f(x) - f(y)| \leq \int_x^y 1 \cdot |f'(t)|\,dt \underset{\text{Hölder}}{\leq} (\int_x^y 1^2 dt)^{1/2} \cdot (\int_x^y |f'(t)|^2 dt)^{1/2} \leq \sqrt{|x-y|} \cdot \sqrt{k_2}$

Same when $x > y$

So $\forall x, y \in [0,1]$: $|f(x) - f(y)| \leq \sqrt{|x-y|} \cdot \sqrt{k_2}$

Given $\varepsilon > 0$: take $\delta = \frac{\varepsilon^2}{k_2} \Rightarrow \forall f \in D$: if $|x - y| < \delta \Rightarrow |f(x) - f(y)| \leq \sqrt{|x-y|} \cdot \sqrt{k_2} < \sqrt{\frac{\varepsilon^2}{k_2}} \cdot \sqrt{k_2} = \varepsilon$.

Baire Category Theorem :

Baire Category thm: X a <u>complete</u> metric space

Let $\{O_n\}_{n=1}^{\infty}$ be a countable collection of open & dense sets in X:

Then $\bigcap_{n=1}^{\infty} O_n$ is dense in X. i.e. $\forall O \subset X$ nonempty : $(\bigcap_{n=1}^{\infty} O_n) \cap O \neq \phi$.

Proof = Recall : $S \subset X$ is dense $\Leftrightarrow X = cl(S) \Leftrightarrow \forall O \subset X$ nonempty & open : $O \cap S \neq \phi$.

Goal : $\forall O \subset X$ nonempty & open : $(\bigcap_{n=1}^{\infty} O_n) \cap O \neq \phi$.

<u>Step 1</u> : Consider $O_1 \cap O$:

- O_1 dense in $X \Rightarrow O_1 \cap O \neq \phi$; also : $\left.\begin{matrix} O \text{ open} \\ O_1 \text{ open} \end{matrix}\right\} \Rightarrow O_1 \cap O$ is open
- pick $x_1 \in O_1 \cap O$: Let $\varepsilon > 0$ ST $B(x_1, \varepsilon) \subset O_1 \cap O \Rightarrow r_1 = \min(\frac{\varepsilon}{2}, 1)$

so $cl(B(x_1, r_1)) \subset \{y : d(x_1, y) \leq \frac{\varepsilon}{2}\} \subset B(x_1, \varepsilon) \subset O_1 \cap O$

$\Rightarrow \exists x_1 \in X$ and $0 < r_1 \leq 1$ ST : $cl(B(x_1, r_1)) \subset O_1 \cap O$

<u>Step 2</u> : Consider $O_2 \cap B(x_1, r_1)$:

- nonempty & open $\Rightarrow$ pick $x_2 \in O_2 \cap B(x_1, r_1)$ and let $\varepsilon > 0$ ST $B(x_2, \varepsilon) \subset O_2 \cap B(x_1, r_1) \Rightarrow r_2 := \min\{\frac{\varepsilon}{2}, \frac{1}{2}\}$
- $cl(B(x_2, r_2)) \subset \{y : d(x_2, y) \leq \frac{\varepsilon}{2}\} \subset B(x_2, \varepsilon) \subset O_2 \cap B(x_1, r_1)$

$\Rightarrow \exists x_2 \in X$ and $0 < r_1 \leq \frac{1}{2}$ ST : $cl(B(x_2, r_2)) \subset O_2 \cap B(x_1, r_1)$

continue inductively :

$\Rightarrow$ Construct a seq : $(x_n)_{n=1}^{\infty} \in X$, $(r_n)_{n=1}^{\infty} \in \mathbb{R}$ ST $0 < r_n \leq \frac{1}{n}$ and : $\begin{cases} n = 1 : cl(B(x_1, r_1)) \subset O_1 \cap O \\ n \geq 2 : cl(B(x_n, r_n)) \subset O_n \cap B(x_{n-1}, r_{n-1}) \end{cases}$

<u>claim</u> : $(x_n)_{n=1}^{\infty} \in X$ is Cauchy : So X complete $\Rightarrow (x_n)_{n=1}^{\infty}$ Converges $\Rightarrow \lim_{n \to \infty} x_n = x \in X$ exists

(pf) Let $\varepsilon > 0$: choose $N > 0$ ST $\frac{2}{N} < \varepsilon$

$\forall n, m \geq N$: $\begin{cases} B(x_m, r_m) \subset B(x_N, r_N) \\ B(x_n, r_n) \subset B(x_N, r_N) \end{cases}$ By construction, so $x_m, x_n \in B(x_N, r_N)$ $\forall n, m \geq N$.

$\Rightarrow d(x_m, x_n) \leq d(x_m, x_N) + d(x_N, x_n) \leq r_N + r_N = \frac{2}{N} < \varepsilon$.

<u>claim</u> : $x \in cl(B(x_n, r_n))$ $\forall n \in \mathbb{N}$.

(pf) fix n : $\forall m > n$: $B(x_m, r_m) \subset B(x_n, r_n) \subset cl(B(x_n, r_n))$

so $(x_m)_{m=n}^{\infty} \in cl(B(x_n, r_n))$ so $\lim_{m \to \infty} x_m = x \in cl(B(x_n, r_n))$ since $cl(B(x_n, r_n))$ is closed and $(x_m)_{m=n}^{\infty}$ converges.

<u>CCL</u> : So $\forall n \geq 2$: $cl(B(x_n, r_n)) \subset O_n \cap B(x_{n-1}, r_{n-1}) \Rightarrow x \in O_n$ $\forall n \geq 2$

$n = 1$: $x \in cl(B(x_1, r_1)) \subset O_1 \cap O \Rightarrow x \in O_1$

$\Rightarrow$ $x \in O_n$ $\forall n \in \mathbb{N}$ <u>AND</u> $x \in O$ $\Rightarrow$ $x \in (\bigcap_{n=1}^{\infty} O_n) \cap O$

$\Rightarrow (\bigcap_{n=1}^{\infty} O_n) \cap O \neq \phi$.

⚠ BCT fails when X is <u>not</u> complete :

<u>EX</u> : $X = \mathbb{Q}$, $d(x, y) = |x - y|$ $\rightarrow \mathbb{Q} = \{r_1, r_2, \dots\}$ enumerat° of rationals.

$V_n := \mathbb{Q} \setminus \{r_n\} \Rightarrow V_n$: open & dense in $\mathbb{Q}$.

so $r_n \notin \bigcap_{n=1}^{\infty} V_n$ $\forall n \in \mathbb{N} \Rightarrow \bigcap_{n=1}^{\infty} V_n = \phi$ while $(V_n)_{n=1}^{\infty}$ is a countable seq of open dense sets in $\mathbb{Q}$ $\rightarrow$ <u>NOT</u> complete!

useful!

BCT corr: X a complete metric space: $\{F_n\}_{n=1}^{\infty}$ a countable set of closed sets in X.

If $\mathrm{Int}\left(\bigcup_{n=1}^{\infty} F_n\right) \neq \phi \Rightarrow \mathrm{Int}(F_n) \neq \phi$ for at least one $n \in \mathbb{N}$.

Proof: Suppose not: let $\{F_n\}_{n=1}^{\infty}$ be closed ST $\mathrm{Int}(F_n) = \phi \ \forall n \in \mathbb{N}$ But $\mathrm{Int}\left(\bigcup_{n=1}^{\infty} F_n\right) \neq \phi$..

- $O_n = F_n^c \Rightarrow O_n$ open & dense (WHY)
- Let V be open & nonempty: then $V \cap O_n \neq \phi$ (o'wise: $V \subset O_n^c = F_n \Rightarrow \mathrm{Int}(F_n) \neq \phi \Rightarrow \Leftarrow$)
- $\mathrm{Int}\left(\bigcup_{n=1}^{\infty} F_n\right) \neq 0 \Rightarrow \exists O \neq \phi$ open ST $O \subset \bigcup_{n=1}^{\infty} F_n \Rightarrow \phi = O \cap \left(\bigcup_{n=1}^{\infty} F_n^c\right) = O \cap \left(\bigcap_{n=1}^{\infty} F_n^c\right) = O \cap \underbrace{\left(\bigcap_{n=1}^{\infty} O_n\right)}_{\text{dense in } X}$

So $O \cap \left(\bigcap_{n=1}^{\infty} O_n\right) = \phi$ but BCT $\Rightarrow O \cap \left(\bigcap_{n=1}^{\infty} O_n\right) \neq \phi$ $\boxed{\Rightarrow\Leftarrow}$

Category:
- A set $A \subset X$ is "nowhere dense" if $\mathrm{Int}(\mathrm{cl}(A)) = \phi$
- $A \subset X$ is of "1st category" if $A = \bigcup_{i=1}^{\infty} B_i$, B_i nowhere dense sets
- otherwise A is of "2nd category"

BCT $\Rightarrow$ A complete metric space X is of 2nd category

Proof: if $X = \bigcup_{n=1}^{\infty} A_n$, A_n nowhere dense $\Rightarrow X = \bigcup_{n=1}^{\infty} \mathrm{cl}(A_n)$

$X \neq \phi \Rightarrow \mathrm{int}\, X \neq \phi \Rightarrow \phi \neq \mathrm{int}\, X = \mathrm{int}\left(\bigcup_{n=1}^{\infty} \mathrm{cl}(A_n)\right) \Rightarrow$ corr: $\exists n_0 \in \mathbb{N}$ ST $\mathrm{Int}(\mathrm{cl}(A_{n_0})) \neq \phi$

$\boxed{\Rightarrow\Leftarrow}$ Since A_{n_0} is nowhere dense.

G_δ set: $A \subset X$ is a G_δ set if: A is a countable intersection of open sets.

Ex: $x \in X$: $O_n = B(x, \frac{1}{n}) \Rightarrow \bigcap_{n=1}^{\infty} O_n = \{x\} \rightarrow$ Any point of X is G_δ.

Ex: $\mathbb{R}^N$

F_σ set: A countable union of closed sets.

Prop: X a complete metric space: $\{V_n\}_{n=1}^{\infty}$ a countable collec^o of dense G_δ sets

then $\bigcap_{n=1}^{\infty} V_n$ is a dense G_δ set

Proof: Let $\{V_n\}_{n=1}^{\infty}$ be a countable collec^o of dense G_δ sets:

V_n is $G_\delta \Rightarrow \exists O_{n_m}$ $(m = 1, 2, \dots)$ open ST $V_n = \bigcap_{m=1}^{\infty} O_{n_m}$ } So $\bigcap_{n=1}^{\infty} V_n = \bigcap_{n=1}^{\infty}\left(\bigcap_{m=1}^{\infty} O_{n_m}\right) = \bigcap_{n,m=1}^{\infty} O_{n_m}$

$V_n \subset O_{n_m}$ and V_n dense $\Rightarrow O_{n_m}$ dense

Since $\{O_{n_m}\}_{n,m=1}^{\infty}$ is a countable coll^o of open sets

$\Rightarrow \bigcap_{n=1}^{\infty} V_n$ is G_δ & dense by BCT

Isolated pt: X a metric space: $x \in X$ is isolated if:

$\exists \epsilon > 0$ ST $B(x, \epsilon) \cap X = \{x\}$

Ex: X any set, d: discrete metric $\rightarrow$ Any point $x \in X$ is isolated (∵ take $\epsilon = 1$: $B(x, 1) = \{y \in X : d(x,y) < 1\} = \{x\}$)

Prop: X a complete metric space: $S \subset X$ countable & dense $\Rightarrow S$ is not G_δ

BCT conseq: $\{O_n\}_{n=1}^{\infty}$ open & dense in $X \Rightarrow \bigcap_{n=1}^{\infty} O_n$ is G_δ and dense

so if X has no isolated pts: $\bigcap_{n=1}^{\infty} O_n$ is uncountable!

Proof: Suppose not: let $S = \{x_1, \dots, x_n, \dots\}$ be a countable G_δ set $\Rightarrow S = \bigcap_{n=1}^{\infty} V_n$ where $\{V_n\}_{n=1}^{\infty}$ are open in X $\Rightarrow$ set $W_n = V_n \setminus \{x_n\}$

Claim: W_n is nonempty:

if $W_n = \phi \Rightarrow V_n = \{x_n\}$: V_n is open $\Rightarrow \exists \epsilon > 0$ ST $B(x_n, \epsilon) = \{x_n\} \Rightarrow \{x_n\}$ is an isolated pt $\boxed{\Rightarrow\Leftarrow}$

Claim: W_n is open:

Let $x \in W_n \Rightarrow x \in V_n$ and $x \neq x_n$: V_n open $\Rightarrow \exists \epsilon > 0$ ST $B(x, \epsilon) \subset V_n$: let $\epsilon' = \min\{\epsilon, d(x, x_n)\} > 0$ (as $d(x, x_n) > 0$)

then $x_n \notin B(x, \epsilon')$ and $B(x, \epsilon') \subset B(x, \epsilon) \subset V_n \Rightarrow B(x, \epsilon') \subset W_n \Rightarrow W_n$ open!

Claim: W_n is dense:

suppose not:

CCL: W_n are open dense sets & $\bigcap_{n=1}^{\infty} W_n \subset \bigcap_{n=1}^{\infty} V_n = S$: $x_n \notin W_n \ \forall n \Rightarrow x_k \notin \bigcap_{n=1}^{\infty} W_n \ \forall k \Rightarrow S \cap \left(\bigcap_{n=1}^{\infty} W_n\right) = \phi$ and $\bigcap_{n=1}^{\infty} W_n \subset S$

$\Rightarrow \bigcap_{n=1}^{\infty} W_n = \phi$ $\boxed{\Rightarrow\Leftarrow}$ By BCT

Consequences of Baire's Category theorem:

① Singular functions:

$(C([a,b]), \|\cdot\|_\infty)$: $f \in C([a,b])$ ST f is not differentiable at any pt of (a,b). → f is "singular"

Prop: BCT $\Rightarrow$ $\exists$ a G_δ set $E \subset C([a,b])$ that consists of singular funct°.

Ex: physics: quasi periodic systems, systems far from equilibrium
Probability theory, modern theory of dynamical systems.

② Fourier Series:

Let $f \in C([-\pi,\pi])$ ST $f(-\pi) = f(\pi)$

Let $f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x)\,dx$ $\rightsquigarrow$ $\sum_{n=-\infty}^{\infty} |f_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$

Let $S_{n,f}(x) = \sum_{k=-n}^{n} f_k(x) e^{ikx}$

Q: when does: $\lim_{n\to\infty} S_{n,f}(x) = f(x)$ $\forall x \in [-\pi,\pi]$?

Prop: f Lipschitz $\Rightarrow$ $S_{n,f}(x) \longrightarrow f(x)$ uniformly on $[-\pi,\pi]$

(e.g.) f Lipschitz $\Rightarrow$ $\exists k>0, \alpha \in (0,1]$ ST: $|f(x)-f(y)| \leq k \cdot |x-y|^\alpha$ $\forall x,y \in [-\pi,\pi]$

then $\lim_{n\to\infty} \max_{x\in[-\pi,\pi]} |S_{n,f}(x) - f(x)| = 0$ (proof in tutorial)

⚠ Fails in general:

Prop: BCT $\Rightarrow$ $\exists$ a dense G_δ set $E \subset C_p([-\pi,\pi])$ ST: (cont f° ST $f(-\pi)=f(\pi)$)

$\forall f \in E$: $\{x \in [-\pi,\pi] : \lim_{n\to\infty} S_{n,f}(x) = \infty\}$ is a G_δ set in $[-\pi,\pi]$

③ Functional Analysis:

3 Fundamental theorems: 2 of them follow from BCT

1) Banach-Steinhaus thm:

X a Banach space, Y a normed space, $\{A_\alpha\}_{\alpha\in I}$ a family of continuous linear maps: $A_\alpha : X \to Y$

then: • Either: $\exists M>0$ ST: $\forall \alpha \in I$, $\|A_\alpha\| = \sup_{\|x\|\leq 1} \|A_\alpha x\| \leq M$

• Or: $\exists$ a dense G_δ set E in X ST: $\sup_{\alpha\in I} \|A_\alpha x\| = \infty$ $\forall x \in E$.

2) Open mapping thm:

X, Y Banach Spaces: $A : X \to Y$ a cont lin map

Then $\forall O \subset X$ open: $A(O) = \{Ax : x \in O\}$ is open in Y.

3) Hahn-Banach extension principle:

Banach Fixed pt thm / Contract° principle :

I) BFP theorem :

Contraction: X a metric space. $f : X \to X$ is a contraction if :
$\exists\, \alpha \in (0,1)$ ST : $d(f(x), f(y)) \leq \alpha\, d(x,y) \quad \forall x, y \in X$

Prop: f is a contraction $\Rightarrow$ f is Lipschitz $\Rightarrow$ f is unif. cont.

BFP thm: X a complete metric space, $f : X \to X$ a contraction
then $\exists!\, x \in X$ ST $f(x) = x$ $\rightsquigarrow$ f has a unique fixed pt !

Proof:
- **uniqueness:** Let $x, y \in X$ ST : $f(x) = x$, $f(y) = y$: $d(x,y) = d(f(x), f(y)) \leq \alpha\, d(x,y)$
$\Rightarrow (1-\alpha)\, d(x,y) \leq 0 \Rightarrow \begin{cases} d(x,y) \leq 0 : \text{but } d(x,y) \geq 0 \Rightarrow d(x,y) = 0 \\ 1 - \alpha \leq 0 : \text{not possible since } 0 < \alpha < 1 \end{cases}$ $d(x,y) = 0 \Rightarrow x = y$.
- **Existence:** pick $x_0 \in X$: define $x_n = f(x_{n-1}) = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}(x_0)$ $\to$ defines $(x_n)_{n=1}^{\infty}$ in X.

Claim: $(x_n)_{n=1}^{\infty}$ is Cauchy :
$\forall n, m, N \in \mathbb{N}$: Suppose $n, m \geq N$ (WLOG : $n > m$)
$d(x_n, x_m) \leq d(x_n, x_{n-1}) + \cdots + d(x_{m+1}, x_m)$
$d(x_n, x_{n-1}) = d(f(x_{n-1}), f(x_{n-2})) \leq \alpha\, d(x_{n-1}, x_{n-2}) = \alpha \cdot d(f(x_{n-2}), f(x_{n-3})) \leq \alpha^2 d(x_{n-2}, x_{n-3}) \leq \cdots$
So $d(x_n, x_{n-1}) \leq \alpha^{n-1} d(x_1, x_0)$
$\Rightarrow d(x_n, x_m) \leq (\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m)\, d(x_1, x_0) = \alpha^m (1 + \alpha + \cdots + \alpha^{n-1-m})\, d(x_1, x_0) = \alpha^m \left(\sum_{k=0}^{n-1-m} \alpha^k\right) d(x_1, x_0)$
$\leq \alpha^m d(x_1, x_0) \sum_{k=0}^{\infty} \alpha^k = \frac{\alpha^m}{1-\alpha} d(x_1, x_0) \leq \frac{\alpha^N}{1-\alpha} d(x_1, x_0)$ as $\alpha \in (0,1)$, $m \geq N$.

So : $d(x_n, x_m) \leq \frac{\alpha^N}{1-\alpha} d(x_1, x_0) \quad \forall n, m \geq N$.
$\lim_{N \to \infty} \alpha^N = 0 \Rightarrow$ Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ ST : $\frac{\alpha^N}{1-\alpha} d(x_1, x_0) < \varepsilon \to d(x_n, x_m) < \varepsilon \quad \forall n, m \in \mathbb{N}$

CCL: $(x_n)_{n=1}^{\infty}$ is Cauchy, X is complete $\Rightarrow \lim_{n \to \infty} x_n = x \in X$ exists
$x_{n+1} = f(x_n) \to x = \lim_{n \to \infty}(x_{n+1}) = \lim_{n \to \infty}(f(x_n)) = f(x)$ as f cont on X. $\Rightarrow x = f(x)$. fixed pt $\exists$

Rate of conv: $d(x, x_N) \leq \frac{\alpha^N}{1-\alpha} d(x_1, x_0)$ at the N^{th} step.
(P) $\forall n \geq N$: $d(x_n, x_N) \leq \frac{\alpha^N}{1-\alpha} d(x_1, x_0)$: $|d(x_n, x_N) - d(x, x_N)| < d(x_n, x) \Rightarrow \lim_{n \to \infty} d(x_n, x_N) = d(x, x_N)$
So $d(x, x_N) \leq \lim_{n \to \infty} d(x_n, x_N) \leq \frac{\alpha^N}{1-\alpha} d(x_1, x_0)$

⚠ X **must** be **complete** !
Ex : $X = (0,1]$ not complete in $(\mathbb{R}, |\,|)$ $\rightsquigarrow$ $f(x) = \frac{x}{2}$: $|f(x) - f(y)| = \frac{1}{2}|x - y|$ is a contract°.
But $f(x) \neq x \quad \forall x \in (0,1]$ ...

Brower Fixed pt thm: $K = \{x \in \mathbb{R}^N : \|x\|_2 \leq 1\}$ the closed unit ball in $\mathbb{R}^N$ $\left(\|x\|_2 = \sqrt{\sum_{k=1}^{N} x_k^2}\right)$
$f : K \to K$ continuous $\Longrightarrow \exists\, x \in K$ ST $f(x) = x$ (may not be unique !)

II Applications to Integral Equations:

Note: The BFPthm can be applied to ODEs for Picard Iterations.

Setup 1: On a closed & bounded interval $[a,b]$, given:

- Integral Kernel: $K : [a,b]\times[a,b] \longrightarrow \mathbb{R}$ continuous
- Source term: G $[a,b] \longrightarrow \mathbb{R}$ continuous
- Control parameter: $\lambda \in \mathbb{R}$, $\lambda \neq 0$
- Unknown function: $f : [a,b] \longrightarrow \mathbb{R}$ continuous

Solve: $\forall x \in [a,b]$

$$\boxed{f(x) = \lambda \int_a^b K(x,y) f(y)\, dy + g(x)}$$

Goal: Find f and discuss the uniqueness of the Solution.

Prop: $H(x) = \int_a^b K(x,y) f(y)\, dy$: $H : [a,b] \longrightarrow \mathbb{R}$ is continuous

Proof: $|H(x) - H(y)| \leq \int_a^b |K(x,y) - K(x',y)| \cdot |f(y)|\, dy \leq \|f\|_\infty \cdot \int_a^b |K(x,y) - K(x',y)|\, dy$ where $\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$

- $K : [a,b]\times[a,b] \longrightarrow \mathbb{R}$ cont, $[a,b]\times[a,b]$ compact $\Rightarrow$ Given $\varepsilon > 0$, $\exists \delta > 0$ ST: $\sqrt{(x-x')^2 + (y-y')^2} < \delta \Rightarrow |K(x,y) - K(x',y')| < \frac{\varepsilon}{\|f\|_\infty (b-a)}$
  In particular: take $y = y'$: $|x - x'| < \delta \Rightarrow |K(x,y) - K(x',y)| < \frac{\varepsilon}{\|f\|_\infty (b-a)}$
- So Given $\varepsilon > 0$, $\exists \delta > 0$ ST: if $|x - x'| < \delta \Longrightarrow |H(x) - H(y)| < \|f\|_\infty \cdot \int_a^b \frac{\varepsilon}{\|f\|_\infty (b-a)}\, dy = \varepsilon$

Def: In $(C([a,b]), \|\cdot\|_\infty)$: let $A : C([a,b]) \longrightarrow C([a,b])$, $(Af)(x) = \int_a^b K(x,y) f(y)\, dy$ $\forall x \in [a,b]$, $\forall f \in C([a,b])$

Prop:
- A is linear: $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ $\forall \alpha, \beta \in \mathbb{R}$, $\forall f, g \in C([a,b])$
- If $M = \max_{(x,y)\in[a,b]\times[a,b]} |K(x,y)| < \infty$ (Since cont on compact) : $\|A(f)\| \leq M \cdot \|f\|_\infty \cdot (b-a)$
- A is Lipschitz (So unif. cont) on $C([a,b])$: $\|A(f) - A(g)\| \leq M \cdot \|f - g\|_\infty \cdot (b-a)$ $\forall f, g \in C([a,b])$

Proof:
- By Linearity of the integral
- $|(Af)(x)| = |\int_a^b K(x,y) f(y)\, dy| \leq \int_a^b |K(x,y)| \cdot |f(y)|\, dy \leq \int_a^b M \cdot \|f\|_\infty\, dy \Rightarrow |(Af)(x)| \leq M \cdot \|f\|_\infty \cdot (b-a)$ $\forall x \in [a,b] \Rightarrow \|A(f)\| \leq M \|f\|_\infty \cdot (b-a)$
- $\|A(f) - A(g)\| = \|A(f-g)\| \leq M \cdot \|f - g\|_\infty \cdot (b-a) \rightarrow A$ Lipschitz $\Rightarrow A$ unif cont!

Rewrite your equation as: $\boxed{f = \lambda A(f) + g}$ and let $\begin{cases} G : C([a,b]) \longrightarrow C([a,b]) \\ G(f) = \lambda A(f) + g \end{cases}$ $\rightsquigarrow$ Goal: BFP for $G(f) = f$!

Prop: Let $\alpha := |\lambda| \cdot M(b-a)$: $0 < \alpha < 1 \Longrightarrow G(f) = f$ has a unique solution

Proof: $\|G(f_1) - G(f_2)\| = \|\lambda A(f_1 - f_2)\| \leq |\lambda| \cdot \|A(f_1 - f_2)\| \leq |\lambda| \cdot M \cdot (b-a) \cdot \|f_1 - f_2\|_\infty = \alpha \|f_1 - f_2\|_\infty$
So $|\alpha| < 1 \Rightarrow \|G(f_1) - G(f_2)\| < \alpha \|f_1 - f_2\|_\infty \Rightarrow G$ is a contraction : $(C([a,b]), \|\cdot\|_\infty)$ Banach $\overset{BFP}{\Longrightarrow} G(f) = f$ has a unique Sol° $\left(|\alpha| < 1 \Leftrightarrow |\lambda| < \frac{1}{M(b-a)}\right)$

Note: Construct the Sol° up to error: Let $f_0(x) = 1$ $\forall x \in [a,b]$, $G^n(f_0) = G \circ G \circ \dots G_0(f_0)$ n-times
If f is the Sol°: $\|f - G^N(f_0)\|_\infty \leq \frac{\alpha^N}{1-\alpha} \|G(f_0) - f_0\|_\infty$

Ex: Let $[a,b] = [0,1]$, $K(x,y) = 1$: $f(x) = \lambda \int_0^1 f(y)\, dy + g(x)$ and $\alpha = |\lambda|$

- $|\lambda| < 1$: BFP $\Rightarrow \exists!$ Sol° : $f = c_\lambda(g) + g$ where $c_\lambda(g) = \frac{\lambda}{1-\lambda} \int_0^1 g(y)\, dy$
- $\begin{cases} |\lambda| > 1 \\ \lambda = -1 \end{cases}$: ~~BFP~~ But $\exists!$ Sol° : $f = c_\lambda(g) + g$ where $c_\lambda(g) = \frac{\lambda}{1-\lambda} \int_0^1 g(y)\, dy$
- $\lambda = 1$ and:
  - $\int_0^1 g(x)\, dx \neq 0$: $\nexists$ Solut°
  - $\int_0^1 g(x)\, dx = 0$: $\exists$ ∞ many Solut° : $f = c + g$, $c \in \mathbb{R}$ arbitrary

(PS)
- $\underline{\lambda \neq -1}$: $f(x) = \lambda \int_0^1 f(y)\, dy + g(x) \Rightarrow \int_0^1 f(x)\, dx = \lambda \int_0^1 f(y)\, dy + \int_0^1 g(x)\, dx \Rightarrow (1-\lambda) \int_0^1 f(y)\, dy = \int_0^1 g(x)\, dx \Rightarrow \int_0^1 f(x)\, dx = \frac{1}{1-\lambda} \int_0^1 g(x)\, dx$
  So f is a Sol° $\Rightarrow f(x) = \lambda \int_0^1 f(y)\, dy + g(x) = \frac{\lambda}{1-\lambda} \int_0^1 g(y)\, dy + g(x) \rightarrow$ UNIQUE!
- $\underline{\lambda = 1}$: $f(x) = \int_0^1 f(y)\, dy + g(x) \Rightarrow \int_0^1 f(x)\, dx = \int_0^1 f(x)\, dx + \int_0^1 g(x)\, dx \Rightarrow \int_0^1 g(x)\, dx = 0$ $\rightarrow$ if true: ∞ many Sol° $f = c + g$: $c + g = f = \int_0^1 (c+g) + g = c + 0 + g$ ✓ ($\int g = 0$)
  $\rightarrow$ if wrong: $\nexists$ Sol°

Compact linear map: $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ normed spaces.

A linear continuous map $\Phi : X \longrightarrow Y$ is compact if $cl\{Af : \|f\|_X \leq 1\} \subset Y$ is compact in $(Y, \|\cdot\|_Y)$.

Prop: on $C([a,b])$: $A(f) = \int_a^b K(x,y) f(y)\,dy$, $A : C([a,b]) \to C([a,b])$ is a compact linear map / compact operator.

Proof:
- $B = \{f \in C([a,b]) : \|f\|_\infty \leq 1\}$ closed unit ball: $\dim(C([a,b])) = \infty \Rightarrow B$ is closed, bounded but NOT compact
- A(B) is bounded: $\|f\|_\infty \leq 1 \Rightarrow \|A(f)\| \leq M(b-a) \cdot \|f\|_\infty \leq M(b-a) \Rightarrow \|A(f)\| \leq M(b-a)\ \forall f \in B \Rightarrow A(B)$ is bounded
- A(B) is equicont: $|(Af)(x) - (Af)(x')| = \left|\int_a^b [K(x,y)f(y) - K(x',y)f(y)]\,dy\right| \leq \int_a^b |K(x,y) - K(x',y)| \cdot \underbrace{|f(y)|}_{\leq 1}\,dy \leq \int_a^b |K(x,y) - K(x',y)|\,dy$

  K unif cont on $[a,b] \Rightarrow \forall \varepsilon > 0, \exists \delta > 0$ ST: $\forall x, x' \in [a,b] : |x - x'| < \delta \Rightarrow |K(x,y) - K(x',y)| < \frac{\varepsilon}{(b-a)}$

  So $\forall \varepsilon > 0, \exists \delta > 0$ ST: $|x - x'| < \delta \Rightarrow |(Af)(x) - (Af)(x')| < \int_a^b \frac{\varepsilon}{b-a}\,dx = \varepsilon$

Arzela-Ascoli $\Rightarrow cl(A(B))$ is compact in $(C([a,b]), \|\cdot\|_\infty) \Rightarrow A$ is a compact operator.

Note: Compact operators are ∞ dimensional, but are "as close as possible" to finite matrices.

II Generalize BFP theorem:

⚠ To apply BFP thm, f must be a contraction: $d(f(x), f(y)) < d(x,y)\ \forall x \neq y$ is NOT enough!

Ex: $X = [0, \infty)$, $d(x,y) = |x - y|$, $f(x) = x + \frac{1}{1+x}$

- $|f(x) - f(y)| = \left|x - y + \frac{1}{1+x} - \frac{1}{1+y}\right| = \left|x - y - \frac{x-y}{(x+1)(y+1)}\right| = |x-y| \cdot \left|1 - \frac{1}{(x+1)(y+1)}\right| < |x - y|$ if $x \neq y$ on $[0, \infty)$
- So $|f(x) - f(y)| < |x - y|\ \forall x \neq y$ BUT f does not have a fixed pt: $f(x) = x \Leftrightarrow x = x + \frac{1}{1+x} \Leftrightarrow \frac{1}{1+x} = 0$ $\boxed{\Rightarrow \Leftarrow}$

Thm: X compact, $d(f(x), f(y)) < d(x,y)\ \forall x \neq y$: then $\exists!\, x \in X$ ST $f(x) = x$.

Proof: HW 5

Thm: X Complete: if $f : X \to X$ is ST: $f^k = \overbrace{f \circ f \circ \dots \circ f}^{k \text{ times}}$, $\exists!\, x \in X$ ST $f^k(x) = x$ for some $k \in \mathbb{N}$, then $\exists!\, x \in X$ ST $f(x) = x$.

Corr: if f^k is a contraction for some $k \in \mathbb{N}$, then $\exists!\, x \in X$ ST $f(x) = x$.

Proof: Let $x \in X$ be ST: $f^k(x) = x$: then $f(f^k(x)) = f(x) \Rightarrow f^{k+1}(x) = f(x) \Rightarrow f^k(f(x)) = f(x) \Rightarrow f(x)$ is a fixed pt of f^k

- By uniqueness of fixed pts of f^k: $f(x) = x \Rightarrow x$ is a fixed pt of f.
- if $f(x) = x$, $f(y) = y$ then $f^k(x) = x$ and $f^k(y) = y \Rightarrow$ By uniqueness of fixed pts of f^k: $x = y$.

Prop: $(X, \|\cdot\|_\infty)$ a Banach Space, $A : X \to X$ a continuous linear map.
if $\exists k \in \mathbb{N}$ ST $A^k = A \circ A \circ \dots \circ A$ is a contraction, then the eq $x = A(x) + y$ has a unique sol^{n} $\forall y \in X$.

Proof: fix $y \in X$: let $f : X \to X$ be defined by $f(x) = A(x) + y$

$f(x) = A(x) + y$

$f^2(x) = A^2x + Ay + y$

$f^3(x) = A^3x + A^2y + Ay + y$

$\vdots$

$f^k(x) = A^kx + A^{k-1}y + \dots + Ay + y$

$\Big\}\ \|f^k(x) - f^k(x')\|_\infty = \|A^kx - A^kx'\|_\infty$

So A^k is a contract^{n} $\Rightarrow f^k$ is a contraction $\Rightarrow f^k$ has a unique fixed point

Then by previous thm: f has a unique fixed pt $\Rightarrow \exists!\, x \in X$ ST $x = f(x)$

ST $x = f(x) = A(x) + y$, $\forall y \in X$

IV Another class of Integral Equations :

<u>Setup 2</u> : on a closed & bounded interval $[a,b]$, given :

- Integral kernel : $k : [a,b]\times[a,b] \longrightarrow \mathbb{R}$ continuous
- Source term : $g : [a,b] \longrightarrow \mathbb{R}$ continuous
- Control parameter : $\lambda \in \mathbb{R}$, $\lambda \neq 0$
- Unknown function : $f : [a,b] \longrightarrow \mathbb{R}$ continuous

Solve : $\forall x \in [a,b]$

$$\boxed{f(x) = \lambda \int_a^x k(x,y) f(y)\,dy + g(x)}$$

<u>Goal</u> : Find f and discuss the uniqueness of the solution.

<u>Def</u> : In $(C([a,b]), \|\cdot\|_\infty)$: let $A : C([a,b]) \longrightarrow C([a,b])$, $(Af)(x) = \int_a^x k(x,y) f(y)\,dy$, $\forall x \in [a,b]$, $\forall f \in C([a,b])$

<u>Prop</u> :
- A is linear
- f continuous $\Rightarrow A(f)$ is continuous on $[a,b]$
- A is Lipschitz : $\|(Af_1)(x) - (Af_2)(x)\|_\infty \leq M \cdot \|f_1 - f_2\|_\infty \cdot (b-a)$, $M = \max_{(x,y)\in[a,b]\times[a,b]} |k(x,y)|$

<u>Proof</u> :
- By linearity of the integral
- $|(Af)(x) - (Af)(x')| = \left| \int_a^x k(x,y) f(y)\,dy - \int_a^{x'} k(x',y) f(y)\,dy \right| \overset{x<x'}{=} \left| \int_a^x [k(x,y) - k(x',y)] f(y)\,dy - \int_x^{x'} k(x',y) f(y)\,dy \right|$

$\leq \int_a^x |k(x,y) - k(x',y)| \cdot |f(y)|\,dy + \int_x^{x'} |k(x',y)| \cdot |f(y)|\,dy$

$\leq \int_a^b \frac{\varepsilon}{2\|f\|_\infty (b-a)} \|f\|_\infty\,dy + |x'-x| \cdot M \cdot \|f\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2M\|f\|_\infty} \cdot M \cdot \|f\|_\infty = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

- $|A(f_1)(x) - (Af_2)(x)| \leq \int_a^b |k(x,y)| \cdot |f_1(y) - f_2(y)|\,dy \leq \|f_1 - f_2\|_\infty \cdot \int_a^b |k(x,y)|\,dy \leq \|f_1 - f_2\|_\infty \cdot (b-a) M$

<u>Prop</u> : If $|\lambda| < \frac{1}{M(b-a)}$: BFP $\Rightarrow \exists! f \in C([a,b])$ ST $f = \lambda A(f) + g$

<u>Proof</u> : $G(f) = \lambda A(f) + g$: $\|G(f_1) - G(f_2)\| = \left\|\lambda \int_a^x k(x,y) f_1(y)\,dy - \int_a^x k(x,y) f_2(y)\,dy\right\| \leq |\lambda| \cdot M \cdot \|f_1 - f_2\|_\infty (b-a)$ $\Rightarrow$ BFP works when $|\lambda| < \frac{1}{M \cdot (b-a)}$

<u>Prop</u> : $f = \lambda \cdot A(f) + g$ <u>has</u> a <u>unique</u> solut° $\forall \lambda \in \mathbb{R}$

<u>Proof</u> : $(Af)(x) = \int_a^x k(x,y) f(y)\,dy$

$(A^2 f)(x) = \int_a^x k(x,y) (A(f))(y)\,dy = \int_a^x k(x,y) \left[\int_a^y k(y,t) f(t)\,dt\right] dy = \int_a^x \int_a^y k(x,y) k(y,t) f(t)\,dt\,dy$

$(A^k(f))(x) = \int_a^x \int_a^{x_1} \cdots \int_a^{x_{k-1}} k(x,x_1) k(x_1,x_2) \cdots k(x_{k-1},y) f(y)\,dy\,dx_{k-1} \cdots dx_2\,dx_1$

So : $|(A^k(f))(x)| \leq \|f\|_\infty \cdot M^k \int_a^b \int_a^{x_1} \cdots \int_a^{x_{k-1}} dy\,dx_{k-1} \cdots dx_1 \leq \|f\|_\infty \cdot M^k \cdot \frac{(b-a)^k}{k!}$

So $|A^k f| \leq \|f\|_\infty \cdot \frac{(M(b-a))^k}{k!}$

So $\|A^k f_1 - A^k f_2\|_\infty \leq \|f_1 - f_2\|_\infty \cdot \boxed{\frac{(M(b-a))^k}{k!}} \longrightarrow 0$ as $k \to \infty$

(*) $e^\alpha = \sum \frac{\alpha^n}{n!} \Rightarrow \frac{\alpha^n}{n!} \to 0$

Hence for large enough k : $\frac{(M(b-a))^k}{k!} < 1$

So A^k is a contract° for k large enough $\Rightarrow f = \lambda Af + g$ has a unique fixed pt $\forall \lambda \in \mathbb{R}$.

(**) $\int_a^{x_{k-1}} dy = (x_{k-1} - a)$

$\int_a^{x_{k-2}} (x_{k-1} - a)\,dx_{k-1} = \frac{(x_{k-2} - a)^2}{2}$

$\int_a^{x_{k-3}} \frac{(x_{k-2} - a)^2}{2}\,dx_{k-2} = \frac{(x_{k-2} - a)^3}{2 \cdot 3}$

$\int_a^b \frac{(x_1 - a)^{k-1}}{(k-1)!}\,dx_1 = \frac{(b-a)^k}{k!}$

TOPOLOGY

Topological Spaces : Basic Definitions

I Open sets, bases and Subbases:

<u>Topology</u> = X a set, τ a collection of subsets of X:
τ is a topology for X if: (1) $\phi, X \in \tau$
(2) τ is closed under arbitrary union : $\{O_i\}_{i\in I} \in \tau \Rightarrow \bigcup_{i\in I} O_i \in \tau$
(3) τ is closed under <u>finite</u> intersection : $O_1, \dots, O_n \in \tau \Rightarrow \bigcap_{i=1}^{n} O_i \in \tau$

<u>open sets</u> = Elements of τ are called open sets

<u>Neighborhood</u> = if $x \in X$: any open set $O_x \in \tau$ containing x is a neighborhood of x.

<u>Ex</u>: • Metric topology : (X,d) a metric space $\Rightarrow$ the collect° τ of all <u>open sets</u> in X is a topology for X.
↳ $O \in X$ is open if $\forall x \in O, \exists \varepsilon_x$ ST $B(x, \varepsilon_x) \subset O$
↳ Euclidean topology : topology on $\mathbb{R}^n$ induced by the Euclidean metric

⚠ Not every topology is metrizable ! → <u>Ex</u>: Temperd distribut° in PDEs : $\delta : C(\mathbb{R}) \longrightarrow \mathbb{R}$
$f \longmapsto \delta(f) = f(0)$
$\delta'(f) = -f'(0)$
→ <u>Ex</u>: Zarinski topology in nb theory

• Trivial topology : $X \neq \phi$: $\tau = \{\phi, X\}$ is a topology for X.
$\forall x \in X$, the only neighborhood of x is X.

• Discrete topology : $X \neq \phi$: $\tau = \{$collect° of all subsets of $X\}$ is a topology for X. → induced by discrete metric !
Every set containing a pt is a neighborhood of that point.

<u>Base for topology at x</u> : (X, τ) a topological space and $x \in X$:
A collect° of neighborhoods $\mathcal{B}_x$ of x are a base for topology at x if : $\forall$ neighborhood O of x $\exists B \in \mathcal{B}_x$ ST $B \subset O$

<u>Base for topology τ</u> : A family of open sets $\mathcal{B}$ in x is a base for topology τ if $\mathcal{B}$ contains a base for a topology $\forall x \in X$.
i.e. $\forall x \in X$, $\forall$ neighborhood O of x : $\exists B \in \mathcal{B}$ ST $x \in B \subset O$.

<u>Prop</u> = $\mathcal{B}$ a collect° of open sets in a topological space (X, τ) :
then: $\mathcal{B}$ is a base for a top τ $\Longleftrightarrow$ Any nonempty open set in τ can be written as a union of some subcollect° of $\mathcal{B}$

<u>Proof</u> = ($\Rightarrow$) Let $O \neq \phi$ and let $x \in O$: then O is a neighborhood of x.
$\mathcal{B}$ is a Base for the topology at x $\Rightarrow \exists B_x \in \mathcal{B}$ ST $x \in B_x \subset O$ $\Rightarrow O \subset \bigcup_{x\in O} B_x \subset O \Rightarrow O = \bigcup_{x\in O} B_x$
($\Leftarrow$) Let $x \in X$ be given : <u>Goal</u> : $\mathcal{B}$ contains a base for the top at x, i.e. : if $O \neq \phi$, O open and $x \in O$, then $\exists B \in \mathcal{B}$ ST $x \in B \subset O$.
O is open $\Rightarrow O = \bigcup_{i\in I} B_i$ for some family $\{B_i\}_{i\in I}$ in $\mathcal{B}$
But if $x \in O$ then $\exists i_0 \in I$ ST : $x \in B_{i_0}$ $\Rightarrow x \in B_{i_0} \subset O$

<u>Conseq</u> = A base uniquely determines a topology : i.e. if τ_1 & τ_2 are two topologies with same base $\mathcal{B} \Rightarrow \tau_1 = \tau_2$

<u>Proof</u> : • if $O \in \tau_1 \Rightarrow O = \bigcup_{i\in I} B_i$ where $B_i \in \mathcal{B}$; $B_i \in \tau_2 \Rightarrow O \in \tau_2$.
• if $O \in \tau_2 \Rightarrow O = \bigcup_{i\in I} B_i$ where $B_i \in \mathcal{B}$; $B_i \in \tau_1 \Rightarrow O \in \tau_1$
$\Big\}$ then $\tau_1 = \tau_2$.

⚠ A given topology can have different basis !

<u>Ex</u>: $\mathbb{R}$, $d(x,y) = |x-y|$ is the metric topology
• One base : Collection of all open intervals with rational nb at the endpts
• Another base : Collection of all open intervals with irrational nb at the endpts

Prop: X a nonempty set, $\mathcal{B}$ a collect° of subsets of X:

$\mathcal{B}$ is a base for a topology for $X \overset{(1)\ (2)}{\iff} \mathcal{B}$ covers X : $X = \bigcup_{B \in \mathcal{B}} B$, and $\forall B_1, B_2 \in \mathcal{B}$ and $\forall x \in B_1 \cap B_2$: $\exists B \in \mathcal{B}$ ST: $x \in B \subset B_1 \cap B_2$.

Proof: ($\Rightarrow$) Let $\mathcal{B}$ be a basis for topology τ in X:

if $x \in X$, X open, then $\exists B_x \in \mathcal{B}$ ST $x \in B_x \subset X \Rightarrow \bigcup_{x \in X} B_x = X \Rightarrow \bigcup_{B \in \mathcal{B}} B = X$

if $B_1, B_2 \in \mathcal{B}$: $x \in B_1 \cap B_2 \Rightarrow B_1 \cap B_2$ is a neighborhood of x (1/open)

$\mathcal{B}$ is a topology at $x \Rightarrow \exists B_x \in \mathcal{B}$ ST $x \in B_x \subset B_1 \cap B_2$

($\Leftarrow$) Let τ be the collect° of all unions of subcoll° of $\mathcal{B}$ together with ϕ

So $A \in \tau \Rightarrow A = \phi$ or $A = \bigcup_{i \in I} B_i$ where $\{B_i\}$ is a collect° of sets in $\mathcal{B}$.

Goal: τ is a topology for X:

- $\phi \in \tau$, and $\mathcal{B}$ covers $X \Rightarrow X \in \tau$.
- τ is closed under arbitrary union:

  (1) Let $\{A_j\}_{j \in J}$ a collect° of sets in τ indexed by set J.

  Then each A_j is a union of some subcoll° of $\mathcal{B}$: $A_j = \bigcup_{i \in I_j} B(i,j)$, $B(i,j) \in \mathcal{B}$.

  So $\bigcup_{j \in J} A_j = \bigcup_{j \in J} \bigcup_{i \in I_j} B(i,j)$ is the union of a family of sets in $\mathcal{B} \Rightarrow \tau$ closed under arbitrary union
- τ is closed under finite intersect°: Enough to show that $\forall A_1, A_2 \in \tau$: $A_1 \cap A_2 \in \tau$.

  (2) Let $A_1 = \bigcup_{i \in I} B_i$, $A_2 = \bigcup_{j \in J} B_j$, $B_i, B_j \in \mathcal{B} \Rightarrow A_1 \cap A_2 = \bigcup_{i \in I, j \in J} (B_i \cap B_j)$

  Let $x \in A_1 \cap A_2$: Then $\exists i_0, j_0$ ST $x \in B_{i_0} \cap B_{j_0}$, So by (2) : $\exists B_x \in \mathcal{B}$ ST $x \in B_x \subset B_{i_0} \cap B_{j_0}$

  Also, $\left\{ \begin{array}{ll} \bigcup_{x \in A_1 \cap A_2} B_x \supset A_1 \cap A_2 & \text{as } x \in B_x \\ \bigcup_{x \in A_1 \cap A_2} B_x \subset A_1 \cap A_2 & \text{as } B_x \subset B_{i_0} \cap B_{j_0} \subset A_1 \cap A_2 \end{array} \right\} \Rightarrow \bigcup_{x \in A_1 \cap A_2} B_x = A_1 \cap A_2 = \bigcup \{\text{subcoll of } \mathcal{B}\}$

  $\Rightarrow A_1 \cap A_2 \in \tau$.

IV Closed Sets:

Closure point: (X, τ) a topological space, $E \subset X$:

$x \in X$ is a closure pt of $E \iff \forall O_x$ of x : $O_x \cap E \neq \phi$.

closure of E: $Cl(E)$ = collect° of all closure pts of E.

closed set: E is closed $\iff E = Cl(E)$; obviously, $E \subset Cl(E)$, and $A \subset B \Rightarrow Cl(A) \subset Cl(B)$

Prop: $Cl(E)$ is closed ($Cl(E) = Cl(Cl(E))$) ; moreover $Cl(E)$ is the smallest closed set containing E.

Proof:
- Goal: $Cl(Cl(E)) \subset Cl(E)$:

  Let $x \in Cl(Cl(E))$: $\forall O_x$ of x, $O_x \cap Cl(E) \neq \phi \Rightarrow \exists x' \in O_x \cap Cl(E) \Rightarrow \begin{cases} x' \in Cl(E) \\ O_x \text{ neighb of } x' \end{cases} \Rightarrow O_x \cap E \neq \phi$

  So $x \in Cl(Cl(E)) \Rightarrow \forall O_x$ of x : $O_x \cap E \neq \phi \Rightarrow x \in Cl(E)$.
- Let F be a closed set ST $E \subset F$:

  take $x \in F^c$; $Cl(F) = F \Rightarrow x \notin Cl(F) \Rightarrow \exists O_x$ of x ST $O_x \cap F = \phi \overset{E \subset F}{\Rightarrow} O_x \cap E = \phi \Rightarrow x \notin Cl(E) \Rightarrow x \in (Cl(E))^c$

  So if $x \in F^c \Rightarrow x \in (Cl(E))^c$, So $x \in Cl(E) \Rightarrow x \in F$, So $Cl(E) \subset F$.

Prop: (X, τ) a topological space: $E \subset X$ is open $\iff E^c$ is closed.

Proof: ($\Rightarrow$) Suppose E is open:

Let $x \in Cl(E^c) \Rightarrow x \notin E$ because E is open (so it would be a neighborhood of x)

$E \cap E^c = \phi$ So : $x \in Cl(E^c) \Rightarrow x \in E^c$, So $Cl(E^c) \subset E^c$ So $Cl(E^c) = E^c \Rightarrow E^c$ is closed.

($\Leftarrow$) Suppose E^c is closed:

Let $x \in E$ then $x \notin Cl(E^c)$ since $E^c = Cl(E^c)$, So $\exists O_x$ of x ST $O_x \subset E \Rightarrow E = \bigcup_{x \in E} O_x = \bigcup$(open sets)

$\Rightarrow E$ is open.

Prop: (X, τ) a topological space:

- ϕ, X are closed
- if $F_1, \dots, F_n$ are closed sets : $F_1 \cup \dots \cup F_n$ is a closed set.
- if $\{F_i\}_{i \in I}$ is an arbitrary family of closed sets : $\bigcap_{i \in I} F_i$ is closed.

Proof: Take the complement (De Morgan's Laws) and use prop of open sets!

The four Separation Properties:

(I) 4 Separation properties:

(X, τ) a topological space:

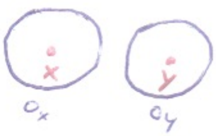
① Tychonoff Separation property: ⟶ too weak to do analysis

Def: Let $x, y \in X$:

$x \neq y \iff \exists\, O_x$ of x and $\exists\, O_y$ of y ST: $x \notin O_y$ and $y \notin O_x$

then (X, τ) is a Tychonoff top space.

② Hausdorff Separation property: ⟶ good enough for analysis



Def: Let $x, y \in X$:

$x \neq y \iff \exists\, O_x$ of x and $\exists\, O_y$ of y ST: $O_x \cap O_y = \phi$.

then (X, τ) is a Hausdorff top space.

③ Regular Separation Property:

Def: Let $x, y \in X$:

$x \neq y \iff X$ is Tychonoff, and $\forall F \subset X$ closed, $x \notin F$: $\exists$ open $V \supset F$ and $\exists\, O_x$ of x ST $V \cap O_x = \phi$.

Then (X, τ) is a regular top Space.

④ Normal Separation property:

Def: Let $x, y \in X$

$x \neq y \iff X$ is Tychonoff, and $\forall$ closed $F_1, F_2 \subset X$, $F_1 \cap F_2 = \phi$: $\exists$ open $O_1 \supset F_1$, $O_2 \supset F_2$ ST $O_1 \cap O_2 = \phi$.

Then (X, τ) is a normal top space.

Prop: Metric $\Rightarrow$ Normal $\Rightarrow$ Regular $\Rightarrow$ Hausdorff $\Rightarrow$ Tychonoff

Metric $\subset$ Normal $\subset$ Regular $\subset$ Hausdorff $\subset$ Tychonoff

Prop: (X, τ) is Tychonoff $\iff$ every set containing only one point is closed.

Proof: ($\Rightarrow$) Let $x \in X$: $\forall y \in X$, $y \neq x \Rightarrow \exists\, O_y$ of y ST $x \notin O_y$

so $\{x\}^c = \bigcup_{y \in X \setminus \{x\}} O_y$ is open (as a union of open sets) $\Rightarrow \{x\}$ is closed

($\Leftarrow$) Fix $x, y \in X$, with $y \neq x$: so $y \in \{x\}^c$ which is open $\Rightarrow \{x\}^c$ is a neighborhood of y that does not contain x.

Also $x \in \{y\}^c$ which is open $\Rightarrow \{y\}^c$ is a neighborhood of x that does not contain y.

so take $O_x = \{y\}^c$, $O_y = \{x\}^c$: $x \in O_x$, $y \in O_y$ but $O_x \cap O_y = \phi$.

Lemma: (X, d) a metric space, $d(x, F) := \inf\{d(x,y) : y \in F\}$ where $F \subset X$, $F \neq \phi$.

- Then F is closed $\Rightarrow$ "$d(x, F) = 0 \iff x \in F$" (*) HW3, #7
- Also, the function $f : x \longmapsto d(x, F)$ is unif. cont on X: (**) HW6, #6

  In fact, f is Lipschitz: $|d(x, F) - d(y, F)| \leq d(x, y)$, i.e. $|f(x) - f(y)| < d(x, y)$

Countability & Separability :

I Definitions :

Countability = • (X, τ) is 1^{st} countable if it has a countable base for a topology at each pt $x \in X$

• (X, τ) is 2^{nd} countable if it has a countable base for a topology τ

Prop = • 2^{nd} countable $\Rightarrow$ 1^{st} countable

• Any metric space is 1^{st} countable. (∵) (X, d) metric space : for each $x \in X$, $\{B(x, \frac{1}{n})\}_{n>0}$ is a countable base for a top at x.

Separability = • a set $S \subset X$ is dense in $X \Leftrightarrow \forall O \subset X$ nonempty : $O \cap S \neq \phi$.

• (X, τ) is separable if $\exists$ a countable dense set $S \subset X$.

Prop = S dense in $X \Leftrightarrow cl(S) = X$

Proof = ($\Rightarrow$) S dense in $X \Rightarrow$ let $x \in X$ and O_x of x : then $O_x \cap S \neq \phi$ $\forall$ neighb O_x of $x \Rightarrow x \in cl(S)$

($\Leftarrow$) $X = cl(S) \Rightarrow$ pick $O \subset X$ nonempty and $x \in O$: then O is a neighb of x and $x \in cl(S) \Rightarrow O \cap S \neq \phi$ for any $O \subset X$.

Prop = • (X, τ) 2^{nd} countable $\Rightarrow$ (X, τ) separable

• (X, τ) separable $\not\Rightarrow$ (X, τ) 2^{nd} countable → Sorgenfrey Line

• (X, d) separable $\Leftrightarrow$ (X, d) 2^{nd} countable

Proof : (1) Let $\{B_n\}_{n=1}^{\infty}$ be a countable basis for a top τ ; pick $x_n \in B_n$ and let $S = \{x_n\}_{n=1}^{\infty}$ → S is countable. S is dense : (∵) Any open set $O \subset X$ nonempty can be written as a union of some sub coll° of $\{B_n\}_{n=1}^{\infty}$ $\Rightarrow O \cap S \neq \phi$

(3) ($\Leftarrow$) if $\{x_n\}_{n=1}^{\infty}$ is a dense set in (X, d), then $\{B(x_n, \frac{1}{m})\}_{n, m \geq 0}$ is a basis for a top for X, which is countable.

($\Rightarrow$) Any open set in (X, d) can be written as a union of some sub coll° of $\{B(x_n, \frac{1}{m})\}_{n, m \geq 0}$ $\Rightarrow$ countable basis for a top for X.

II Metrization :

Metrization = (X, τ) is metrizable $\Leftrightarrow$ $\exists$ a metric d on X ST the topology of open sets WRT d is precisely τ.

Ex = • $X \neq \phi$: Let $\tau = \mathcal{P}(X) = 2^X$ (collect° of all subsets of X) ⇝ τ is the maximal topology.

τ is metrizable : (∵) let $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$ → any point $\{x\}$ is open (as $B(x, \frac{1}{2}) = \{x\}$), so any set is open

⚠ • If X has more than 2 pts : the trivial topology $\tau = \{\phi, X\}$ is NEVER metrizable.

(∵) if d exists : take $x \neq y$ in X, $r = d(x,y) > 0$, $B(x,r) = \{z : d(x,z) < r\}$: $x \in B(x,r)$ but $y \notin B(x,r)$ ($\Rightarrow\Leftarrow$) → d cannot exist

Urysohn Metrization thm = (X, τ) 2^{nd} countable : X metrizable $\Leftrightarrow$ X normal

Characterizat° of closure = • (X, d) metric space, $E \subset X$: $x \in cl(E) \Leftrightarrow \exists$ a seq $(x_n)_{n=1}^{\infty}$ in E ST $\lim_{n \to \infty} x_n = x$.

• (X, τ) 1^{st} countable top space : $x \in cl(E) \Leftrightarrow \exists$ a seq $(x_n)_{n=1}^{\infty}$ in E ST $\lim_{n \to \infty} x_n = x$.

• (X, τ) top space : if $\exists$ a seq $(x_n)_{n=1}^{\infty}$ in E ST $x_n \to x$, then $x \in cl(E)$.

⚠ For top spaces : $x \in cl(E) \not\Rightarrow \exists$ a seq $(x_n)_{n=1}^{\infty}$ in E ST $x_n \to x$

Prop = Any metric space is Normal

Proof = Let (X, d) be a metric space : Let $F_1, F_2 \subset X$ be closed

Goal = Construct open sets $O_1 \supset F_1$, $O_2 \supset F_2$ ST $O_1 \cap O_2 = \phi$

- If $F_1 = \phi$ or $F_2 = \phi$: take respectively $O_1 = \phi$ or $O_2 = \phi$
- If $F_1 \neq \phi, F_2 \neq \phi$: Let $\begin{cases} f(x) = d(x, F_1) - d(x, F_2) \\ g(x) = d(x, F_2) - d(x, F_1) \end{cases} \Rightarrow$ Take : $\begin{cases} O_1 = \{x \in X : f(x) < 0\} = f^{-1}((-\infty, 0)) \\ O_2 = \{x \in X : g(x) < 0\} = g^{-1}((-\infty, 0)) \end{cases}$

- Clearly, $O_1 \cap O_2 = \phi$
- * O_1 & O_2 are open : $(\because)$ Lemma 2 $\Rightarrow$ f, g cont, $(-\infty, 0)$ open $\Rightarrow \begin{cases} O_1 = f^{-1}(\text{open}) \text{ is open} \\ O_2 = g^{-1}(\text{open}) \text{ is open} \end{cases}$
- * $F_1 \subset O_1$: $(\because)$ if $x \in F_1$, $d(x, F_1) = 0$ and $d(x, F_2) > 0$ $\left((\because) F_1 \cap F_2 = \phi \Rightarrow x \notin F_2, \text{ so } F_2 \text{ closed} \xRightarrow{\text{Lemma 1}} d(x, F_2) > 0\right)$ therefore $f(x) = -d(x, F_2) < 0 \Rightarrow x \in O_1 \ \forall x \in F_1$.
- * $F_2 \subset O_2$: Similar argument

II Convergence :

Convergence = (X, τ) a top space : $(x_n)_{n=1}^{\infty}$ a Seq in X

$x_n \to x \in X \iff$ "$\forall O_x$ of x, $\exists N_{O_x} > 0$ ST $x_n \in O_x \ \forall n \geq N_{O_x}$"

Prop = If (X, τ) is Hausdorff : If a sequence has a limit, then the limit is unique.

Proof = $(x_n)_{n=1}^{\infty}$ a seq in X with $\begin{cases} \lim x_n = x \\ \lim x_n = y \end{cases}$ but $x \neq y$.

X hausdorff $\Rightarrow \exists O_x$ of x, O_y of y ST $O_x \cap O_y = \phi$ $\Rightarrow$ Take : $\begin{cases} N_1 \in \mathbb{N} \text{ ST } x_n \in O_x \ \forall n \geq N_1 \\ N_2 \in \mathbb{N} \text{ ST } x_n \in O_y \ \forall n \geq N_2 \end{cases}$, Let $N = \max\{N_1, N_2\}$

Then $x_n \in O_x \cap O_y \ \forall n \geq N \Rightarrow O_x \cap O_y \neq \phi$ $\boxed{\Rightarrow\Leftarrow}$

Ex 1 = $X = \mathbb{R}$, $\tau = \{\mathbb{R}, \phi, (-\infty, c) : c \in \mathbb{R}\}$: $\begin{cases} X \text{ is not Tychonoff} \\ (x_n)_{n=1}^{\infty}, \ x_n = -n \ : \text{ Then } x_n \to x \ \forall x \in \mathbb{R} \text{ (limit NOT unique)} \end{cases}$

Proof =
- τ is a topology for $\mathbb{R}$:
  - $\phi, \mathbb{R} \in \tau$ ✓
  - τ is closed under finite intersec$^{\circ}$: $O_1, \dots, O_n \in \tau \Rightarrow O_1 \cap \dots \cap O_n \in \tau$.
    $(\because)$ Trivial if $O_i = \phi$; if $O_i = \mathbb{R}$ drop it (does not affect $\cap$)
    if $O_i \neq \phi \ \forall 1 \leq i \leq n$: $O_1 \cap \dots \cap O_n = \bigcap_{i=1}^{n} (-\infty, c_i) = (-\infty, c) \in \tau$ where $c = \min\{c_1, \dots, c_n\}$
  - τ is closed under arbitrary union : $\{O_i\}_{i=1}^{\infty} \in \tau \Rightarrow \bigcup O_i \in \tau$.
    $(\because)$ WLOG $O_i \neq \phi$, $O_i \neq \mathbb{R}$, i.e. $O_i = (-\infty, c_i)$
    if $\sup c_i = \infty$: $\forall n \in \mathbb{N}, \exists c_{i_n}$ ST $c_{i_n} > n \Rightarrow (-\infty, n) \subset \bigcup_{i \in I} O_i \ \forall n \in \mathbb{N} \Rightarrow (-\infty, \infty) \subset \bigcup_{i \in I} O_i \subset \mathbb{R} \Rightarrow \bigcup_{i \in I} O_i = \mathbb{R}$
    $\Rightarrow \bigcup_{i \in I} O_i \in \tau$.
    if $\sup c_i = c < \infty$: then $(-\infty, c_i) \subset (-\infty, c) \ \forall i \in I \Rightarrow \bigcup_{i \in I} O_i \subset (-\infty, c)$
    Also : $c - \frac{1}{n}$ is not a UB for $\{c_i\}_{i \in I}$, so $(-\infty, c) = \bigcup_{n=1}^{\infty} (-\infty, c - \frac{1}{n}) \subset \bigcup_{i \in I} O_i$ $\Bigg\} \Rightarrow \bigcup_{i \in I} O_i = (-\infty, c) \in \tau$.
- τ is not Tychonoff :
  $(\because)$ $x, y \in \mathbb{R}$, $x < y$: then any open set containing y is of the form $(-\infty, c)$ where $c > y$.
  So $c > y > x \Rightarrow x \in (-\infty, c)$
- $\lim x_n = x \ \forall x \in \mathbb{R}$ =
  $x_n = -n$: pick any $x \in \mathbb{R}$: then any O_x of x is of the form $O_x = (-\infty, c)$ for $c > x$. Choose N ST $-N < c$
  $\Rightarrow \forall n \geq N : x_n \in O_x$.

Ex 2 = $X = \mathbb{R}$, $\tau = \{\mathbb{R}, \phi, O : O^c \text{ is a finite set}\}$: $\begin{cases} X \text{ is tychonoff but not Hausdorff} \\ (x_n)_{n=1}^{\infty} : x_n = n \Rightarrow x_n \to x \ \forall x \in \mathbb{R} \text{ (limit not unique)} \end{cases}$

Proof =
- τ is a topology for $\mathbb{R}$ =
  - $\phi, \mathbb{R} \in \tau$
  - $\forall O_1, \dots, O_n \in \tau$ (WLOG : $O_i \neq \phi, \mathbb{R}$) : $(O_1 \cap \dots \cap O_n)^c = O_1^c \cup \dots \cup O_n^c$ is finite as O_i^c is finite $\Rightarrow O_1 \cap \dots \cap O_n \in \tau$.
  - $\forall \{O_i\}_{i \in I}$ (WLOG : $O_i \neq \phi, \mathbb{R}$) : $(\bigcup_{i \in I} O_i)^c = \bigcap_{i \in I} O_i^c$ is finite as O_i^c is finite $\Rightarrow \bigcup_{i \in I} O_i \in \tau$.
- τ is Tychonoff =
  Let $x, y \in \mathbb{R}$, $x \neq y$: choose $\begin{cases} O_y = \mathbb{R} \setminus \{x\} \\ O_x = \mathbb{R} \setminus \{y\} \end{cases} \Rightarrow \begin{cases} O_y^c = \{x\} \\ O_x^c = \{y\} \end{cases}$ So $O_x, O_y \in \tau$, $\begin{cases} O_x \text{ neighb of } x, & y \notin O_x \\ O_y \text{ neighb of } y, & x \notin O_y \end{cases}$
- τ is NOT Hausdorff = Let V_x of x, V_y of y, V_x^c & V_y^c finite ST $V_x \cap V_y = \phi \Rightarrow V_x^c \cup V_y^c = \phi^c = \mathbb{R}$ not finite $\boxed{\Rightarrow\Leftarrow}$
- Limit NOT unique = pick any $x \in \mathbb{R}$: let O neighb of x, $O \neq \mathbb{R}, \phi$: So O^c consists of finitely many pts $((\because)$ finite set$)$
  But $x_n = n$ consists of distinct pts $\Rightarrow \exists N > 0$ ST $x_n \notin O^c \ \forall n \geq N \Rightarrow x_n \in O \ \forall n \geq N \Rightarrow \lim x_n = x \ \forall x \in \mathbb{R}$.

Continuous f° on top spaces:

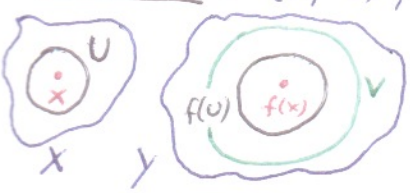
I Continuous functions:

<u>Basic prop</u>: (X, τ) a top space : $V \subset X$ is open $\Leftrightarrow \forall x \in V, \exists O_x$ of x ST $O_x \subset V$

<u>Proof</u>: $\Rightarrow$ V open $\Rightarrow$ take $O_x = V$

$\Leftarrow$ $V = \bigcup_{x \in V} O_x$, O_x open $\Rightarrow V$ open.

<u>cont funct°</u>: $(X, \tau), (Y, S)$ top spaces , $f : X \to Y$ a function



- f cont at $x \in X \Leftrightarrow \forall$ neighb V of $f(x)$, $\exists$ a neighb U of x ST $f(U) \subset V$.
- f cont on $X \Leftrightarrow f$ cont at every $x \in X$.

<u>Prop</u>: $f : X \to Y$ is continuous $\Leftrightarrow \forall$ open $O_Y \subset Y$, $f^{-1}(O_Y) \subset X$ is open.

<u>Proof</u>: $\Rightarrow$ Let $O \subset Y$ open : $f^{-1}(O) = \{x \in X : f(x) \in O\} \to$ let $x \in f^{-1}(O)$.
then $f(x) \in O$ and O is a neighb of $f(x)$: f cont $\Rightarrow \exists$ a neighb $U_x \subset X$ of x ST $f(U_x) \subset O$
$\Rightarrow U_x \subset f^{-1}(O)$
So $\forall x \in f^{-1}(O)$, $\exists_{\text{open}} U_x \subset X$ of x ST $U_x \subset f^{-1}(O) \Rightarrow f^{-1}(O)$ open.

$\Leftarrow$ $\forall$ open $O \subset V$: $f^{-1}(O) \subset X$ is open.
Let $x \in X$, V a neighb of $f(x)$: then V open, $f(x) \subset V \Rightarrow f^{-1}(V)$ open in X, $x \in f^{-1}(V)$.
set $U = f^{-1}(V)$: U is a neighb of x and $f(U) = f(f^{-1}(V)) \in V$ $\to f$ cont at x $\Rightarrow$ Valid $\forall x \in X$, so f cont on X.

<u>Prop</u>: $(X, \tau), (Y, S), (Z, R)$ top spaces : $\begin{cases} f : X \to Y \\ g : Y \to Z \end{cases}$ cont f° $\Rightarrow h = g \circ f : X \to Z$ is cont

<u>Proof</u>: $\forall$ open $O_z \subset Z$: $h^{-1}(O_z) = (g \circ f)^{-1}(O_z) = f^{-1}(g^{-1}(O_z)) = f^{-1}(\underset{\text{in } Y}{\text{open}}) = \underset{\text{in } X}{\text{open}}$

II Smallest topology containing a subset:

<u>Thm</u>: X a set, A = a collect° of subsets of X : (A is usually not a topology)

Let τ_A = collect° of all topologies for X that contain A : ($\tau_A \neq \emptyset$ since $\overset{\text{power set}}{\mathcal{P}(A)} \subset \tau_A$)

then $\tau := \bigcap_{S \in \tau_A} S$ is the minimal topology that contains A $\longrightarrow \tau$ = topology generated by A

<u>Application</u>: $(X_\alpha, \tau_\alpha)_{\alpha \in I}$ a family of topological spaces :

for each α, we are given a $f_\alpha : X \to X_\alpha$

then the topology τ generated by $A = \{f_\alpha^{-1}(O) : O \in \tau_A, \alpha \in I\}$ is the minimal topology in X WRT which, each f_α is continuous

Compact top. Spaces:

I) Compact top. Spaces:

Compactness = (X,τ) is compact if every open cover of X has a finite subcover.

i.e. if $X = \bigcup_{\alpha\in I} V_\alpha$, $V_\alpha \in \tau$, then $\exists\, V_{\alpha_1},\dots,V_{\alpha_n} \in \tau$ ST $X = V_{\alpha_1}\cup\dots\cup V_{\alpha_n}$

Finite intersect° prop = "$\forall$ family $\{F_\alpha\}_{\alpha\in I}$ of closed sets in (X,τ) ST $F_{\alpha_1}\cap\dots\cap F_{\alpha_n} \neq \phi$ $\forall\, \alpha_1,\dots,\alpha_n \in I \Rightarrow \bigcap_{\alpha\in I} F_\alpha \neq \phi$"

then X has finite intersect° property

Prop = (X,τ) compact $\Leftrightarrow$ (X,τ) has finite intersect° prop. $\rightsquigarrow$ same proof as for metric spaces

Compact subsets = A subset Y of (X,τ) is compact if any open cover of Y has a finite subcover.

i.e. if $Y \subset \bigcup_{\alpha\in I} V_\alpha$, $V_\alpha\in\tau$, then $\exists\, V_{\alpha_1},\dots,V_{\alpha_n}$ ST $Y \subset V_{\alpha_1}\cup\dots\cup V_{\alpha_n}$

Prop = (X,τ) compact : $Y\subset X$ closed $\Rightarrow$ Y compact ⚠ Y compact $\not\Rightarrow$ Y closed

Proof = Let $\{V_\alpha\}_{\alpha\in I}$ an open cover of Y : $Y \subset \bigcup_{\alpha\in I} V_\alpha$

Let $O = Y^c = X\setminus Y$ (So O is open) : then $\{O, \{V_\alpha\}_{\alpha\in I}\}$ is an open cover of X : $X = O \cup (\bigcup_{\alpha\in I} O_\alpha)$

So $\exists\, \alpha_1,\dots,\alpha_n \in I$ ST $\begin{cases} X = O\cup(V_{\alpha_1}\cup\dots\cup V_{\alpha_n}) = Y^c \cup (V_{\alpha_1}\cup\dots\cup V_{\alpha_n}) \\ X = Y^c \cup Y \end{cases} \Rightarrow Y \subset V_{\alpha_1}\cup\dots\cup V_{\alpha_n}$

$\Rightarrow Y$ compact.

Ex: $X = \mathbb{R}$, $\tau = \{\phi, \mathbb{R}, O\subset\mathbb{R} : O^c \text{ is a finite set}\}$ $\rightarrow$ τ Tychonoff but not Hausdorff.

Any nontrivial open set $O\in\tau$ is compact in τ! $\longrightarrow$ open sets can be compact!

$\hookrightarrow O \neq \phi, \mathbb{R}$

Prop = (X,τ) compact and Hausdorff : $Y\subset X$ closed $\Leftrightarrow$ Y compact.

Proof = ($\Rightarrow$) done

($\Leftarrow$) Goal = Y^c is open (so Y is closed)

- Let $x\in Y^c$: $\forall y\in Y$, X Hausdorff $\Rightarrow \exists$ Neighborhoods $\begin{cases} V_y \text{ of } y \\ U_y \text{ of } x \end{cases}$ ST $U_y \cap V_y = \phi$.
  $y \neq x$
- $y\in V_y \Rightarrow \{V_y\}_{y\in Y}$ is an open cover of Y : Y compact $\Rightarrow \exists\, y_1,\dots,y_n\in Y$ ST $V_{y_1}\cup\dots\cup V_{y_n} \supset Y$.
- Set $U = U_{y_1}\cup\dots\cup U_{y_n}$: so U is a neighb of x, $U\cap(V_{y_1}\cup\dots\cup V_{y_n}) = \bigcup_{i=1}^{n}\bigcup_{j=1}^{n}(U_{y_i}\cap V_{y_j}) = \phi$

So $U\cap V = \phi$ and $U\subset Y^c$

So : $\forall x\in Y^c$, $\exists$ a neighb U of x ST $U\subset Y^c$ $\Rightarrow$ Y^c open $\rightsquigarrow$ Y closed!

Prop = (X,τ) compact and Hausdorff $\Longrightarrow$ (X,τ) is normal.

Proof = Goal = $F_1, F_2 \subset X$ closed & nonempty : $F_1\cap F_2 = \phi$ $\Rightarrow$ $\exists$ open $\begin{cases} O_1 \supset F_1 \\ O_2\supset F_2 \end{cases}$ ST $O_1\cap O_2 = \phi$

- If $F_1 = \{x\}$: repeat the argument of the previous prop

  X Hausdorff $\Rightarrow \forall y\in F_2, \exists$ neighb $\begin{cases} U_y \text{ of } x \\ V_y \text{ of } y \end{cases}$ ST $U_y\cap V_y = \phi$

  $\{V_y\}_{y\in Y}$ is an open cover of $F_2 \subset X$, F_2 is compact (X compact + Hausdorff, F_2 closed) $\Rightarrow \exists\, y_1,\dots,y_n\in F_2$ ST $F_2 \subset V_{y_1}\cup\dots\cup V_{y_n}$

  Set $\begin{cases} O_1 = U_{y_1}\cap\dots\cap U_{y_n} \rightarrow \text{neighb of } x_1 \text{ so } O_1 \text{ open}, O_1\supset F_1 \\ O_2 = V_{y_1}\cap\dots\cap V_{y_n} \rightarrow \text{open}, O_2\supset F_2 \end{cases}$ & $O_1\cap O_2 = \phi$.

- For a general F_1 : $\forall x\in F_1$, $\exists \begin{cases} U_x \text{ neighb of } x \\ \text{open } V_x \supset F_2 \end{cases}$ ST $U_x\cap V_x = \phi$

  $\{U_x\}_{x\in F_1}$ is an open cover of F_1 $\Rightarrow$ $\exists$ a fin sub cover : $F_1 \subset U_{x_1}\cup\dots\cup U_{x_n}$

  Set : $\begin{cases} O_1 = U_{x_1}\cup\dots\cup U_{x_n} \\ O_2 = V_{x_1}\cup\dots\cup V_{x_n} \end{cases}$ $\Rightarrow$ $\begin{cases} O_1 \text{ open}, O_1\supset F_1 \\ O_2 \text{ open}, O_2\supset F_1 \end{cases}$ and $O_1\cap O_2 = \phi$.

Continuous functions on Compact top Spaces:

Prop: $(X,\tau),(Y,S)$ top spaces, $f: X \to Y$ a cont f°:

X Compact $\Rightarrow f(X) = \{f(x) : x \in X\} \subset Y$ is compact.

Proof: Let $\{V_i\}_{i\in I}$ be an open cover of $f(X)$ in Y ST: $f(X) \subset \bigcup_{i\in I} V_i \Rightarrow X = \bigcup_{i\in I} f^{-1}(V_i)$

So $\{f^{-1}(V_i)\}_{i\in I}$ is an open cover of $X \Rightarrow X = f^{-1}(V_{i_1}) \cup \dots \cup f^{-1}(V_{i_n})$

So $f(X) = f(f^{-1}(V_{i_1}) \cup \dots \cup f^{-1}(V_{i_n})) \subset V_{i_1} \cup \dots \cup V_{i_n} \Rightarrow$ fin. Subcover of $f(X)$

Prop: (X,τ) Compact, $f: X \to \mathbb{R}$ cont $\Rightarrow f$ achieves its min & max on X.

Proof: $f(X) \subset \mathbb{R}$ is compact, so $f(X)$ is closed & bounded (as $\mathbb{R}$ is Hausdorff, $X \subset \mathbb{R}$)
So $f(X)$ contains its inf & sup.

Note:
- (X,d) a metric space: Compactness $\Leftrightarrow$ Seq Compactness
- (X,τ) a 2^{nd} Countable Top Space: Compactness $\Leftrightarrow$ Seq Compactness
- (X,τ) a general top space: Compactness $\not\Leftrightarrow$ Seq Compactness
  - $\hookrightarrow$ need notion of nets! ("every net has a conv subnet")

Connected topological Spaces:

I) Definition & basic Properties:

Connected: • (X,τ) is connected if it cannot be written as a union of 2 disjoint nonempty open sets.

i.e. X connected $\Leftrightarrow$ " $X = O_1 \cup O_2$, O_1 & O_2 open, $O_1 \cap O_2 = \phi \Rightarrow O_1 = \phi$ or $O_2 = \phi$ "

• A subset Y of (X,τ) is connected $\Leftrightarrow$ " $Y \subset O_1 \cup O_2$, O_1 & O_2 open, $Y \cap O_1 \cap O_2 = \phi \Rightarrow \begin{cases} Y \cap O_1 = \phi \\ \text{or} \\ Y \cap O_2 = \phi \end{cases}$ "

Clarification: (X,τ), $Y \subset X$: Then the family $S = \{Y \cap O : O \in \tau\}$ is a topology for Y.
(Y,S) is connected $\Leftrightarrow$ $Y \subset X$ is connected (WRT τ). $\hookrightarrow (Y,S)$ is a "Top. Subsp. of (X,τ)"

Prop: (X,τ) is connected $\Leftrightarrow$ ϕ and X are the only 2 subsets of X that are both open & closed.

Proof: $(\Rightarrow)$ Let A open & A^c open: then $\begin{cases} A \cup A^c = X \\ A \cap A^c = \phi \\ A, A^c \text{ open} \end{cases} \xRightarrow{X \text{ connected}} A = \phi$ or $A^c = \phi \Rightarrow A = \phi$ or X.

$(\Leftarrow)$ Let $\begin{cases} O_1 \cup O_2 = X \\ O_1 \cap O_2 = \phi \\ O_1, O_2 \text{ open} \end{cases} \Rightarrow O_1 = O_2^c$, so O_1 is both open & closed $\Rightarrow$ either $O_1 = \phi$ or $O_1 = X$
$\Rightarrow$ either $O_1 = \phi$ or $O_2 = \phi$
$\Rightarrow X$ is connected.

Prop: (X,τ), (Y,S) Top spaces, $f: X \to Y$ a cont f°
X connected $\Rightarrow f(X) = \{f(x) : x \in X\} \subset Y$ is connected

Proof: suppose that: $\exists V_1, V_2$ open ST: $f(X) \subset V_1 \cup V_2$, $f(X) \cap V_1 \cap V_2 = \phi$:

• $\begin{cases} X = f^{-1}(V_1) \cup f^{-1}(V_2) \\ \phi = f^{-1}(V_1) \cap f^{-1}(V_2) \end{cases}$ (:) o/wise if $x \in f^{-1}(V_1) \cap f^{-1}(V_2)$, then $\begin{cases} f(x) \in V_1 \\ f(x) \in V_2 \end{cases} \Rightarrow f(x) \cap V_1 \cap V_2 \neq \phi$ $\boxed{\Rightarrow\Leftarrow}$

• X is connected $\Rightarrow$ either $\begin{cases} f^{-1}(V_1) = \phi \Leftrightarrow f(X) \cap V_1 = \phi \\ \text{OR} \\ f^{-1}(V_2) = \phi \Leftrightarrow f(X) \cap V_2 = \phi \end{cases} \Rightarrow f(X)$ is connected!

Intermediate value thm: (X,τ) connected, $f: X \to \mathbb{R}$ a cont f°, $a = \inf(f(X))$, $b = \sup(f(X))$:
then if $a < r < b$, then $\exists x \in X$ ST $f(x) = r$

Proof: Suppose not: then $\exists r \in (a,b)$ ST $f(x) \neq r$ $\forall x \in X$
Set: $\begin{cases} O_1 = (-\infty, r) \\ O_2 = (r, \infty) \end{cases} \Rightarrow \begin{cases} f(X) \subset O_1 \cup O_2 \\ f(X) \cap \underbrace{O_1 \cap O_2}_{=\phi} = \phi \end{cases}$

• $f(X) \cap O_1 \neq \phi$: (:) $r > a = \inf(f(X)) \Rightarrow \exists x_r \in X$ ST $r > f(x_r) \Rightarrow f(x_r) \in f(X) \cap O_1 \Rightarrow f(X) \cap O_1 \neq \phi$.
• $f(X) \cap O_2 \neq \phi$: (:) $r < b = \sup(f(X)) \Rightarrow \exists x_r \in X$ ST $r < f(x_r) \Rightarrow f(x_r) \in f(X) \cap O_2 \Rightarrow f(X) \cap O_2 \neq \phi$.
$\Rightarrow f(X) \subset \mathbb{R}$ cannot be connected $\boxed{\Rightarrow\Leftarrow}$ since X connected, f cont so $f(X)$ connected!

Useful prop: (X,τ), $Y = \{0,1\}$ with discrete metric (every subset is open)

Then: X is connected $\Leftrightarrow$ Every continuous funct° $f: X \to Y$ is CST (i.e. $f(x) = \begin{cases} 0, \forall x \in X \\ \text{or} \\ 1, \forall x \in X \end{cases}$)

Proof: $(\Rightarrow)$ Let f be a cont f°: $f: X \to Y$, let $\begin{cases} O_1 = f^{-1}(\{0\}) \\ O_2 = f^{-1}(\{1\}) \end{cases}$: $\{0\}, \{1\}$ are open in Y $\Rightarrow O_1, O_2$ are open in X
So $X = O_1 \cup O_2$ and $O_1 \cap O_2 = \phi$
X is connected $\Rightarrow$ either $O_1 = \phi$ or $O_2 = \phi \Rightarrow$ either $f(x) = 0$ $\forall x \in X$ or $f(x) = 1$ $\forall x \in X$.

$(\Leftarrow)$ Goal: prove contrapositive: " X not connected $\Rightarrow \exists$ a cont f° $f: X \to Y$ which is NOT CST "
X not connected $\Rightarrow \exists O_1, O_2$ open in X ST: $\begin{cases} X = O_1 \cup O_2 \\ \phi = O_1 \cap O_2 \end{cases}$ but $O_1 \neq \phi$, $O_2 \neq \phi$.

define: $f(x) = \begin{cases} 1, & x \in O_1 \\ 0, & x \in O_2 \end{cases}$ $\Rightarrow$ • f is not CST as $O_1 \neq \phi$ and $O_2 \neq \phi$.
• f is continuous since $f^{-1}\{1\} \to O_1$ & $f^{-1}\{0\} \to O_2$ (open, open; open, open)

Prop: $(X_1, d_1), (X_2, d_2)$ metric spaces, $\begin{cases} d((x_1,x_2),(x_1',x_2')) = \sqrt{(d_1(x_1,x_1'))^2 + (d_2(x_2,x_2'))^2} \\ X = X_1 \times X_2 \end{cases}$

X is connected $\Longleftrightarrow$ X_1 & X_2 are connected.

Proof: $(\Rightarrow)$ Let $\begin{cases} \pi_1(x_1,x_2) = x_1 \\ \pi_2(x_1,x_2) = x_2 \end{cases}$ so $\begin{cases} \pi_1 : X \xrightarrow{X_1 \times X_2} X_1 \\ \pi_2 : X \longrightarrow X_2 \end{cases}$ are both continuous, so $\begin{cases} X_1 = \pi_1(X) \\ X_2 = \pi_2(X) \end{cases}$: X connected $\Rightarrow X_1, X_2$ connected

$(\Leftarrow)$ Let $f : X \to Y = \{0,1\}$ a cont f° $\longrightarrow$ Goal: $f = CST$

- given $x_1 \in X_1$: set $\begin{cases} h_{x_1} : X_2 \longrightarrow \{0,1\} \\ h_{x_1}(x_2) = f(x_1,x_2) \end{cases}$, so f cont $\Rightarrow h_{x_1}$ is cont (as f is cont, x_1 fixed); so X_2 connected $\Rightarrow h_{x_1} = CST$.
- given $x_2 \in X_2$: set $\begin{cases} h_{x_2} : X_1 \longrightarrow \{0,1\} \\ h_{x_2}(x_1) = f(x_1,x_2) \end{cases}$, so f cont $\Rightarrow h_{x_2}$ is cont; so X_1 connected $\Rightarrow h_{x_2} = CST$.
- If $(x_1,x_2), (x_1',x_2') \in X = X_1 \times X_2$: $f(x_1,x_2) = h_{x_1}(x_2) \underset{h_1 = CST}{=} h_{x_1}(x_2') = f(x_1,x_2') = h_{x_2'}(x_1) \underset{h_2 = CST}{=} h_{x_2'}(x_1') = f(x_1',x_2')$

So $f(x_1,x_2) = f(x_1',x_2') \Rightarrow f = CST$ on X.

Prop: (X_n, d_n), $n = 1,2,\dots$ metric spaces, $X = \prod_{n=1}^{\infty} X_n$: X connected $\Longleftrightarrow$ every X_n is connected.

Proof: HW 6, #1

Prop: (X, τ) a top. space, $\{A_i\}_{i \in I}$ an arbitrary collect° of connected subsets of X.
If : $\forall i,j \in I$, $A_i \cap A_j \neq \phi$, then $A = \bigcup_{i \in I} A_i$ is connected.

Proof: Suppose A is <u>not</u> connected : then $\exists\, O_1, O_2 \subset X$ open ST : $\begin{cases} A \subset O_1 \cup O_2 \\ A \cap O_1 \cap O_2 = \phi \end{cases}$ And $\begin{cases} A \cap O_1 \neq \phi \\ A \cap O_2 \neq \phi \end{cases}$

- But $\phi \neq A \cap O_1 = (\bigcup_{i \in I} A_i) \cap O_1 = \bigcup_{i \in I} (A_i \cap O_1) \Rightarrow A_i \cap O_1 \neq \phi$ for some $i \in I$
  But $\phi \neq A \cap O_2 = (\bigcup_{i \in I} A_i) \cap O_2 = \bigcup_{i \in I} (A_i \cap O_2) \Rightarrow A_i \cap O_2 \neq \phi$ for some $i \in I$
  $\Rightarrow \exists (i,j)$ in I ST $\begin{cases} A_i \cap O_1 \neq \phi \\ A_j \cap O_2 \neq \phi \end{cases}$
- Also, $A \subset O_1 \cup O_2 \Rightarrow \begin{cases} A_i \subset O_1 \cup O_2 \\ A_j \subset O_1 \cup O_2 \end{cases}$; Also, $A \cap O_1 \cap O_2 = \phi \Rightarrow \begin{cases} A_i \cap O_1 \cap O_2 = \phi \\ A_j \cap O_1 \cap O_2 = \phi \end{cases}$

  Since A_i, A_j are connected, then $\begin{cases} A_i \subset O_1 \\ A_j \subset O_2 \end{cases}$ and $\begin{cases} A_i \cap O_2 = \phi \\ A_j \cap O_1 = \phi \end{cases}$ $\Rightarrow A_i \cap A_j = A_i \cap A_j \cap O_1 \cap O_2 \subset \underbrace{A \cap O_1 \cap O_2}_{= \phi}$
  $\Rightarrow A_i \cap A_j = \phi$ $\boxed{\Rightarrow \Leftarrow}$

Prop: (X, τ), $A \subset X$: A connected $\Rightarrow Cl(A)$ is also connected ⚠ $Cl(A)$ connected $\not\Rightarrow$ A connected

Proof: HW 6, #3.1

(II) Connected Components :

Connected Component : A subset C of (X, τ) is a "Connected Component" if : • C is connected
• $\forall$ connected subset $D \subset X$ ST $C \subset D$ then $C = D$
↳ "maximal connected subset of X"

Prop = Any connected component C is a closed set. (∵) C is a connected comp $\Rightarrow$ C is connected $\Rightarrow$ $cl(C)$ is connected. But $C \subset cl(C)$, so by def : $C = cl(C)$.

Prop = C_1, C_2 connected comp : then either $C_1 = C_2$ or $C_1 \cap C_2 = \phi$

(∵) if $C_1 \cap C_2 \neq \phi$: C_1, C_2 connected $\Rightarrow$ $C_1 \cup C_2$ connected (as $C_1 \cap C_2 \neq \phi$) : $\begin{cases} C_1 \subset C_1 \cup C_2 \\ C_2 \subset C_1 \cup C_2 \end{cases} \Rightarrow C_1 = C_1 \cup C_2 = C_2 \Rightarrow C_1 = C_2$.

Prop = (X, τ) : Any point $x \in X$ is contained in some connected component of X.

Lemma = Singlets $\{x\}$ are connected (∵) if $\begin{cases} \{x\} \subset O_1 \cup O_2 \\ \{x\} \cap O_1 \cap O_2 = \phi \end{cases}$ then either $\begin{cases} \{x\} \cap O_1 = \phi \\ \text{or} \\ \{x\} \cap O_2 = \phi \end{cases}$ (o'wise $O_1 \cap O_2 = \{x\}$)

Proof = Let A = collect° of all connected sets containing a given $x \in X$. ⇝ $\forall A_i \in A : \{x\} \in A_i$
then $A \neq \phi$ since $\{x\} \subset A$.

• Let $C = \bigcup_{D \in A} D$: $\left.\begin{array}{l} \text{every element of } A \text{ is connected} \\ \text{the intersect° of 2 elements of } A \neq \phi \end{array}\right\} \Rightarrow C = \bigcup_{D \in A} D$ is connected, and $\{x\} \in C$

↳ since $\{x\} \in$ this intersect°

• Let $\tilde{C}$ be a connected set ST $C \subset \tilde{C}$
then $\{x\} \in \tilde{C} \Rightarrow \tilde{C} \in A \Rightarrow \tilde{C} \subset C$ $\Big\} \Rightarrow C = \tilde{C}$, so C is a connected component.

Decomposit° of X into connected components : Given (X, τ) : let $\{C_i\}_{i \in I}$ = collect° of all connected comp of X.
then each C_i is closed, disjoint & connected, and $X = \bigcup_{i \in I} C_i$ ⇝ $\{C_i\}_{i \in I}$ is the decomp of X into connected components

⚠ Connected components can be closed but NOT open.

Ex = $X = \mathbb{Q}$, $d(x,y) = |x - y|$:

• claim : if $A \subset \mathbb{Q}$ has 2 or more pts, then A is NOT connected.

(∵) Let $r_1, r_2 \in A$, $r_1 < r_2$: Let $i \in \mathbb{Q}^c$ (irrational) ST $r_1 < i < r_2$, Let $\begin{cases} O_1 = (-\infty, i) = \{x \in \mathbb{Q} : x < i\} \\ O_2 = (i, \infty) = \{x \in \mathbb{Q} : x > i\} \end{cases}$

So $\begin{cases} A \subset O_1 \cup O_2 \\ A \cap O_1 \cap O_2 = \phi \end{cases}$ and $\begin{cases} A \cap O_1 \ni r_1 \neq \phi \\ A \cap O_2 \ni r_2 \neq \phi \end{cases}$ $\Rightarrow$ A is NOT connected

• So the only connected subsets of $\mathbb{Q}$ are singlets $\{r\}$ where $r \in \mathbb{Q}$
So $\mathbb{Q} = \bigcup_{r \in \mathbb{Q}} \{r\}$ is a decomposit° of $\mathbb{Q}$ into connected components
So each component is closed but is not open !

Prop = If (X, τ) has finitely many connected components, then each connected component is both closed and open.

Proof = $C_1, \dots, C_n$ connected comp of X : $X = C_1 \cup \dots \cup C_n$ where the C_i's are closed & disjoint.
Then $C_1 = (C_2 \cup \dots \cup C_n)^c = \underbrace{C_2^c \cap \dots \cap C_n^c}_{\text{open}} = \text{open}$ → valid $\forall C_i$: every C_i is open.
So every connected comp is both open & closed in X

III Examples of Connected Sets :

Ex 1 = Any point in a top. space forms a connected set.

Ex 2 = Sometimes, singlets are the <u>ONLY</u> connected subsets of the top. space. → "Totally disconnected space" !

Ex : $(\mathbb{Q}, |\cdot|)$

Ex : Any set $X \neq \phi$ with $d(x,y) = \begin{cases} 0, & x=y \\ 1, & x \neq y \end{cases}$ → Any set containing 2 or more pts is NOT connected

↳ subsets of X are both open & closed

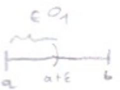
(∵) Let A have more than 2 pts : pick $x \in A$, let $\begin{cases} O_1 = \{x\} \\ O_2 = A \setminus \{x\} \end{cases}$

then $\begin{cases} A \subset O_1 \cup O_2 \\ A \cap O_1 \cap O_2 = \phi \end{cases}$ but $\begin{cases} A \cap O_1 = \{x\} \neq \phi \\ A \cap O_2 \neq \phi \end{cases}$ ⇒ A is not connected

Ex 3 = The Real line :

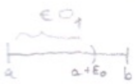
Prop = $-\infty < a < b < \infty$: the closed interval $[a,b]$ is a connected subset of $\mathbb{R}$:

Proof = By contradiction : Suppose $[a,b]$ is NOT connected ⇒ $\exists O_1, O_2$ open ST $\begin{cases} [a,b] \subset O_1 \cup O_2 \\ [a,b] \cap O_1 \cap O_2 = \phi \end{cases}$ but $\begin{cases} [a,b] \cap O_1 \neq \phi \\ [a,b] \cap O_2 \neq \phi \end{cases}$



So either $a \in O_1$ or $a \in O_2$: WLOG $a \in O_1$

• Define $R = \{ \varepsilon : 0 < \varepsilon < b-a$ and $[a, a+\varepsilon) \subset O_1 \}$



→ $R \neq \phi$: (∵) O_1 open, $a \in O_1$ ⇒ $\exists \delta > 0$ ST $(a-\delta, a+\delta) \subset O_1$: take $\delta < b-a$ ⇒ $[a, a+\delta) \subset (a-\delta, a+\delta) \subset O_1$ ⇒ $\delta \in R$.

→ $\sup R = \varepsilon_0 < \infty$: (∵) $0 < \varepsilon < b-a$ ⇒ R is bounded ; $\mathbb{R}$ complete ⇒ R has a sup : $0 < \varepsilon_0 \leq b-a$

• claim 1 : $[a, a+\varepsilon_0) \subset O_1$

(∵) Let $n > 0$ an integer ST $n > \frac{1}{\varepsilon_0}$: So $0 < \varepsilon_0 - \frac{1}{n} < \varepsilon_0$ ⇒ $\varepsilon_0 - \frac{1}{n}$ is <u>not</u> a UB of R ⇒ $\exists \varepsilon' \in R$ ST $\varepsilon_0 - \frac{1}{n} < \varepsilon' < \varepsilon_0$

then $\begin{cases} [a, a+\varepsilon_0 - \frac{1}{n}) \subset [a, a+\varepsilon') \subset [a, a+\varepsilon_0) \\ [a, a+\varepsilon_0 - \frac{1}{n}) \subset [a, a+\varepsilon') \subset O_1 \end{cases}$ ⇒ $[a; a+\varepsilon_0) = \bigcup_{n > \frac{1}{\varepsilon_0}} [a, a+\varepsilon_0 - \frac{1}{n}) \subset O_1$

• claim 2 = $a + \varepsilon_0 \notin O_1$

(∵) Suppose $a + \varepsilon_0 \in O_1$: then either

case 1 : $a + \varepsilon_0 = b$ ⇒ $[a,b] \subset O_1$ ⇒ $[a,b] \cap O_2 = [a,b] \cap O_1 \cap O_2 = \phi$ $\boxed{\Rightarrow\Leftarrow}$ as $[a,b] \cap O_2 \neq \phi$ by def.

OR case 2 = $a + \varepsilon_0 < b$ ⇒ $\exists \delta > 0$ ST $\begin{cases} a+\varepsilon_0+\delta < b \\ (a+\varepsilon_0-\delta, a+\varepsilon_0+\delta) \subset O_1 \text{ open} \end{cases}$ ⇒ $(a, a+\varepsilon_0+\delta) = [a, a+\varepsilon_0) \cup [a+\varepsilon_0, a+\varepsilon_0+\delta) \subset O_1$

⇒ $\varepsilon_0 + \delta \in R$ ... $\boxed{\Rightarrow\Leftarrow}$ as $\varepsilon_0 = \sup R$ and $\varepsilon_0 + \delta > \varepsilon_0$

⇒ $a + \varepsilon_0 \notin O_1$

• claim 3 = $a + \varepsilon_0 \notin O_2$

(∵) Suppose $a+\varepsilon_0 \in O_2$: then $\exists \delta > 0$, $\varepsilon_0 > \delta > 0$, ST $(a+\varepsilon_0-\delta, a+\varepsilon_0-\delta) \subset O_2$

take $r \in (a+\varepsilon_0-\delta, a+\varepsilon_0)$: then $\begin{cases} [a,r) \not\subset O_1 & (∵)\ (a+\varepsilon_0-\delta, r) \subset O_2 \text{ and } O_1 \cap O_2 = \phi \\ [a,r] \subset O_1 & (∵) \begin{cases} r < a+\varepsilon_0 \\ [a, a+\varepsilon_0) \subset O_1 \end{cases} \Rightarrow [a,r] \subset [a, a+\varepsilon_0) \subset O_1 \end{cases}$ $\boxed{\Rightarrow\Leftarrow}$

• Conclusion : $a + \varepsilon_0 \notin O_1$ <u>and</u> $a + \varepsilon_0 \notin O_2$

But $\varepsilon_0 \leq b-a$ ⇒ $a + \varepsilon_0 \in [a,b] \subset O_1 \cup O_2$ $\boxed{\Rightarrow\Leftarrow}$

Prop = • $-\infty \leq a < b \leq \infty$: $(a,b) \subset \mathbb{R}$ is a connected subset (Proof on next page)

• $-\infty \leq a < b < \infty$: $(a,b] \subset \mathbb{R}$ is a connected subset

• $-\infty < a < b \leq \infty$: $[a,b) \subset \mathbb{R}$ is a connected subset

Prop = $O \subset \mathbb{R}$ is a nonempty subset of $\mathbb{R}$: O connected & open ⇔ $O = (a,b)$ for some $-\infty \leq a < b \leq \infty$ (proof on next page)

Corr = $(a,b]$, $[a,b)$ and $[a,b]$ are <u>not</u> open

Prop = $A \subset \mathbb{R}$ a nonempty subset of $\mathbb{R}$: A connected ⇔ $A = \begin{cases} (a,b) \text{ OR} & -\infty \leq a < b \leq \infty \\ (a,b] \text{ OR} & -\infty \leq a < b < \infty \\ [a,b) \text{ OR} & -\infty < a < b \leq \infty \\ [a,b] & -\infty < a < b < \infty \end{cases}$

Proof = • Let $\begin{cases} a = \inf A \\ b = \sup A \end{cases}$ ⇒ $\forall x \in A$: $a \leq x \leq b$ ⇒ $A \subset [a,b]$

• Repeat the arg of the previous prop (next page) to show that : $(a,b) \subset A$

⇒ $(a,b) \subset A \subset [a,b]$ → 4 possibilities for A ! (either $a,b \in A$ or $a,b \notin A$ or $a \notin A$, or $b \notin A$)

Prop = $A_1, \dots, A_n \subset \mathbb{R}$ connected ⇒ $A_1 \times A_2 \times \dots \times A_n \subset \mathbb{R}^n$ is connected

Ex = • $\mathbb{Q} \times \mathbb{Q} \subset \mathbb{R}^2$ is <u>NOT</u> connected ! $\begin{cases} O_1 = \{(x,y) : x,y \in \mathbb{Q}, y < \pi\} \\ O_2 = \{(x,y) : x,y \in \mathbb{Q}, y > \pi\} \end{cases}$ are open and separate $\mathbb{Q} \times \mathbb{Q}$.

• $A = \{(x,y) \in \mathbb{R}^2 : \text{either } x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$: $A \subset \mathbb{R}^2$ is connected. (∵) ↳ or → (use path connectedness)

Prop = $f : [0,1] \to [0,1]$ a cont function ⇒ $\exists x \in [0,1]$ ST $f(x) = x$

Proof = • let $h(x) = f(x) - x$: f cont ⇒ h cont on $[0,1]$, $\begin{cases} h(0) = f(0) - 0 \geq 0 \\ h(1) = f(1) - 1 \leq 0 \end{cases}$ → if $h(0) = 0$ ⇒ $f(0) = 0$ ✓ → if $h(1) = 0$ ⇒ $f(1) = 1$ ✓

• if $\begin{cases} h(1) < 0 \\ h(0) > 0 \end{cases}$ ⇒ $\begin{cases} a = \inf(h[0,1]) < 0 \\ b = \sup(h[0,1]) > 0 \end{cases}$ • $\begin{rcases} [0,1] \text{ connected} \\ h \text{ cont on } [0,1] \end{rcases}$ ⇒ Intermediate Val thm : $a < 0 < b$ ⇒ $\exists x \in (a,b)$ ST $h(x) = 0$ ⇒ $f(x) = x$ ✓

Prop = $-\infty \leq a < b \leq \infty$: $(a,b) \subset \mathbb{R}$ is a connected subset of $\mathbb{R}$.

Proof = • $-\infty < a < b < \infty$: then $(a,b) = \bigcup_{n > \frac{2}{b-a}} [a+\frac{1}{n}, b-\frac{1}{n}]$ → let $A_n = [a+\frac{1}{n}, b-\frac{1}{n}]$ ⇒ A is connected (previous prop)

Also, $A_n \cap A_m \neq \phi$ $\forall n,m > \frac{2}{b-a}$: (∵) $\frac{a+b}{2} \in A_n \cap A_m$ (center of $[a,b]$)

So $(A_n)_{n>\frac{2}{b-a}}^{\infty}$ is a family of non-disjoint connected sets ⇒ $\bigcup_{n>\frac{2}{b-a}} A_n$ is connected.

• $a \neq -\infty, b = \infty$: Take $(a,\infty) = \bigcup_{n=1}^{\infty} [a, a+n] = \bigcup_{n=1}^{\infty} A_n$: $A_n \cap A_m \neq \phi$ as $a \in A_n$ $\forall n \geq 1$, A_n connected ⇒ $\bigcup_{n\geq1}^{\infty} A_n$ connected

• $a = -\infty, b \neq -\infty$: Take $(-\infty, b) = \bigcup_{n=1}^{\infty} [-n, b] = \bigcup_{n=1}^{\infty} A_n$ $(b \in A_n \ \forall n \geq 1)$

• $a = -\infty, b = \infty$: Take $(-\infty,\infty) = \bigcup_{n=1}^{\infty} [-n,n] = \bigcup_{n=1}^{\infty} A_n$ $(0 \in A_n \ \forall n \geq 1)$

Prop = $O \subset \mathbb{R}$ Nonempty open & connected ⇒ $\exists -\infty \leq a < b \leq \infty$ ST $O = (a,b)$ → So $(a,b]$, $[a,b)$, $[a,b]$ are not open!

Proof = Let $\begin{cases} a = \inf O \\ b = \sup O \end{cases}$: Goal : $a, b \notin O$, but $(a,b) \subset O$

→ $a \notin O$: (∵) • if $a = -\infty$ ⇒ $a \notin O$

• if $a \neq -\infty$ and $a \in O$: O open ⇒ $\exists \delta > 0$ ST $(a-\delta, a+\delta) \subset O$ ⇒ $\exists x \in O$ ST $x < a$ (take $x \in (a-\delta, a)$) ... $\boxed{\Rightarrow\Leftarrow}$ as $a = \inf O$

→ $b \notin O$: (∵) • if $b = \infty$ ⇒ $b \notin O$

• if $b \neq \infty$ and $b \in O$: b open ⇒ $\exists \delta > 0$ ST $(b-\delta, b+\delta) \subset O$ ⇒ $\exists x \in O$ ST $x > b$ (take $x \in (b, b+\delta)$) ... $\boxed{\Rightarrow\Leftarrow}$ as $b = \sup O$

→ $O \subset (a,b)$: (∵) $x \in O \Rightarrow a < x < b$ because $a, b \notin O$

→ $(a,b) \subset O$: (∵) Suppose Not : $\exists r \in (a,b)$ ST $r \notin O$ → let $\begin{cases} O_1 = (-\infty, r) \\ O_2 = (r, \infty) \end{cases}$ ⇒ $\begin{cases} O \subset O_1 \cup O_2 \\ O \cap O_1 \cap O_2 = \phi \end{cases}$ (∵) $O_1 \cap O_2 = \phi$

However : $\begin{cases} O \cap O_1 \neq \phi & (∵)\ r > a \text{ and } r \text{ is not a LB for } O \\ O \cap O_2 \neq \phi & (∵)\ r < b \text{ and } r \text{ is not a UB for } O \end{cases}$ ⇒ O is not connected $\boxed{\Rightarrow\Leftarrow}$

IV Path Connectedness :

Path : (X,d) a metric space :

A path in X is a continuous map $f : [0,1] \longrightarrow X$ Such that : $\begin{cases} f(0) = x_0 & : \text{initial pt of the path} \\ f(1) = x_1 & : \text{final pt of the path} \end{cases}$

loop = $f : [0,1] \to X$ a path ST $f(0) = f(1)$

Path-connectedness = $x, y \in X$ are path connected ⟺ ∃ a path $f : [0,1] \to X$ ST $\begin{cases} f(0) = x \\ f(1) = y \end{cases}$

Prop : "$x \sim y$ ⟺ x, y are path connected" is a relat° of equivalence

Proof = • $x \sim x$: (∵) $f(t) = x$ $\forall t \in [0,1]$

• $x \sim y \Rightarrow y \sim x$: (∵) $x \sim y \Rightarrow \exists f : [0,1] \to X$ ST $f(0) = x, f(1) = y$ → take $g(t) = f(1-t)$ ⇒ $g(0) = f(1) = y$, $g(1) = f(0) = x$

• $x \sim y, y \sim z \Rightarrow x \sim z$: (∵) $\begin{cases} \exists f : [0,1] \to X \text{ ST } f \text{ connects } x \text{ to } y \\ \exists g : [0,1] \to X \text{ ST } g \text{ connects } y \text{ to } z \end{cases}$ ⇒ $h(t) = \begin{cases} f(2t) & : 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & : \frac{1}{2} \leq t \leq 1 \end{cases}$ ⇒ h cont on $[0,\frac{1}{2}) \cup (\frac{1}{2},1]$, $h(\frac{1}{2}) = f(1) = g(0)$ ⇒ h cont at $t = \frac{1}{2}$.

Path connectedness = • (X,d) is path connected ⟺ $\forall x, y \in X$, x and y can be connected by a path in X.

• $Y \subset X$ is path connected ⟺ (Y,d) is path connected

⟺ $\forall x, y \in Y$, x and y can be connected by a path that lies entirely in Y.

Thm : (X,d) path connected ⇒ (X,d) connected

Proof = Fix $x \in X$: $\forall y \in X$, ∃ a path $f_y : [0,1] \to X$ connecting x to y : Set $U_y = f_y([0,1]) = \{f_y(t) : t \in [0,1]\}$

f_y cont on $[0,1]$, $[0,1]$ connected ⇒ $U_y \subset X$ connected, $y \in U_y$ ⇒ $\bigcup_{y \in X} U_y = X$

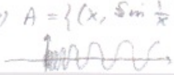
So : $\forall y_1, y_2 \in X$: $x \in U_{y_1} \cap U_{y_2}$ ⇒ $U_{y_1} \cap U_{y_2} \neq \phi$ ⇒ X is connected.

⚠ Converse NOT True in general ! Connected $\not\Rightarrow$ path connected

Ex : • If $X = \mathbb{R}$: $A \subset \mathbb{R}$ connected ⇒ $A \subset \mathbb{R}$ path connected (∵) $A \subset \mathbb{R}$ connected ⇒ $A = (a,b)$, $[a,b]$, $[a,b)$ or $(a,b]$

So take : $f(t) = (1-t)x + ty$ ⇒ path connected

• If $X = \mathbb{R}^2$: Converse fails ! (∵) $A = \{(x, \sin\frac{1}{x}) : 0 < x \leq \frac{2}{\pi}\} \cup \{(0,y) : -1 \leq y \leq 1\} \subset \mathbb{R}^2$ is connected but NOT path connected.



Path Connectedness :

Prop : $(X, \|\cdot\|)$ a normed space :

- The ball of radius $r>0$ centered at x_0 : $B(x_0, r)$ is path connected
- $O \subset X$ open & connected $\Longrightarrow$ O is path connected

Prof :
- $x, y \in B(x_0, r)$: let $f(t) = (1-t)x + ty$ $\forall t \in [0,1]$ $\longrightarrow$ claim: f lies entirely in $B(x_0, r)$

  (∵) $\|f(t) - x_0\| = \|(1-t)x + ty - x_0\| = \|((1-t)x + ty) - ((1-t)x_0 + tx_0)\| \leq (1-t)\|x - x_0\| + t\|y - x_0\| < (1-t)r + tr = r$

- Given $x \in O$, $O_1 = \{y \in O : x \sim y\}$ (with $x \sim y \Leftrightarrow x, y$ path connected) $\Longrightarrow$ O_1 is path connected
  - $O_1 \neq \phi$: (∵) $x \in O_1$ as $x \sim x$
  - O_1 open : (∵) $y \in O_1$, take $r>0$ st $B(y, r) \subset O$: if $z \in B(y, r)$ then $y \sim z$ ($f(t) = (1-t)y + tz$)

    So $x \sim y$ and $y \sim z$ $\Longrightarrow$ $x \sim z$ $\Longrightarrow$ $B(y, r) \subset O_1$ $\Longrightarrow$ O_1 open
  - $O_1 = O$:

    (∵) Suppose NOT : $O_1 \neq O \Longrightarrow \exists y \in O$ st $y \not\sim x$

    Let $O_2 = \{y \in O : y \not\sim x\}$
    - $O_2 \neq \phi$ and $O_2 \cap O_1 = \phi$: the y above is in O_2 ; $O_1 \cap O_2 = \phi$ by def : $x \sim y \Longrightarrow \neg(x \not\sim y)$
    - O_2 is also open : let $y \in O_2$, $r>0$ st $B(y, r) \subset O$:

      then $\forall z \in B(y, r)$ we have $y \sim z$: if $x \sim z \overset{z \sim y}{\Longrightarrow} x \sim y$ ($\Longrightarrow\Longleftarrow$) as $y \in O_2$

      $\Longrightarrow \forall z \in B(y, r) : x \not\sim z \Longrightarrow B(y, r) \subset O_2 \Longrightarrow O_2$ open.
    - $O_2 = \phi$: $\begin{cases} O = O_1 \cup O_2 \\ \phi = O_1 \cap O_2 \end{cases}$ but $\begin{cases} O_1 \neq \phi \\ O_2 \neq \phi \end{cases}$ $\Longrightarrow$ O is not connected $\boxed{\Longrightarrow\Longleftarrow}$

      So our assumpt° that $O_2 \neq \phi$ is false ! $\Longrightarrow$ $O_1 = O$ ✓.

$\Longrightarrow$ $O_1 = O$, So O_1 is path connected $\Longrightarrow$ O is path connected!

Fundamental group: (not on final!

(X, d) a metric space : assume it is path connected.
Fix $x_0 \in X$ and consider the collection of all loops : $\begin{cases} f : [0,1] \to X \\ f(0) = f(1) = x_0 \end{cases}$

Def: we say that 2 loops are equivalent if $\exists$ a cont map:

$$H : [0,1] \times [0,1] \longrightarrow X \quad ST \begin{cases} H(t,0) = f(t) & , \text{for } t \in [0,1] \\ H(0,t) = g(t) & , \text{for } t \in [0,1] \\ H(0,s) = H(1,s) = x_0, & \text{for } s \in [0,1] \end{cases}$$

for any fixed $s \in [0,1]$, the map $t \longmapsto H(t,s)$ is a loop.

$s = 0 \Rightarrow$ the loop is $f(t)$
$s = 1 \Rightarrow$ the loop is $g(t)$
$\Rightarrow$ H is a continuous deformation of loop f to g with X.

Introduce the relation $f \sim g \Leftrightarrow f$ & g are homotopically equiv
$\rightarrow$ this is a relation of equivalence on the set of loops with the same base point x_0.

consider classes of equiv : $[f]$

$[f]^{-1}$ = class of equiv containing $g = f(1-t)$

and $[f] \times [g] = [h]$, where $h(t) = \begin{cases} f(2t) & , 0 \le t \le \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$

The set of all classes of equivalences with these 2 operations is a group

This group is denoted $\pi_1(X)$ and is called the fund group of X.

Ex: S^1 :

$\Rightarrow \pi_1(S^1) = \mathbb{Z}$

$S^1 \times S^1$:

$\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$

$\rightarrow$ these 2 circles cannot be continuously deformed into one another

(X, d_1), (Y, d_2) :

$h : X \to Y$ a bijection, h continuous and h^{-1} cont.

$h_* : \pi_1(X) \to \pi_1(Y)$ isomorphic.

$\mathbb{R}^n, \mathbb{R}^m$ $(n \neq m)$: $\mathbb{R}, \mathbb{R}^n$ $(n > 1)$.