Things to know:

- $\mathrm{Sup}(A+B) \leq \mathrm{Sup}(A) + \mathrm{Sup}(B)$; $\mathrm{Sup}(|a|A) = |a| \,\mathrm{Sup}(A)$

- $G(x)$ cont & increasing f°: $G(\sup_{x\in X} f(x)) = \sup_{x \in X} G(f(x))$ $\forall f: X \longrightarrow [0,\infty)$ bounded.
  $\hookrightarrow G: [0,\infty) \longrightarrow [0,\infty)$

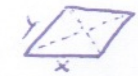
- $\inf(A+B) \geq \inf(A) + \inf(B)$; $\inf(|a|A) = |a| \inf(A)$

- $\mathrm{Sup}(-|a|A) = |a| \inf(A)$

- $A \subset B \Rightarrow \begin{cases} \mathrm{Sup}(A) \leq \mathrm{Sup}(B) \\ \inf(A) \geq \inf(B) \end{cases}$

- $\begin{cases} x^* = \mathrm{Sup}\, A \Rightarrow \forall \varepsilon > 0, \exists a_\varepsilon \in A \text{ ST: } x^* - \varepsilon < a_\varepsilon \rightsquigarrow x^* - \varepsilon \text{ not a UB} \\ x_* = \inf A \Rightarrow \forall \varepsilon > 0, \exists a_\varepsilon \in A \text{ ST: } x_* + \varepsilon > a_\varepsilon \rightsquigarrow x_* + \varepsilon \text{ not a LB} \end{cases}$

Metric Spaces :

- $\langle \cdot,\cdot \rangle \xrightarrow{\|x\| = \sqrt{\langle x,x\rangle}} \|\cdot\| \xrightarrow{d(x,y) = \|x-y\|} d(\cdot,\cdot)$
- Cauchy-Schwartz : $|\langle x,y\rangle| \leq \sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle} \quad \forall x,y \in X$
- Angle between Vectors: $\exists!\, \theta \in [0,\pi]$ ST: $\cos\theta = \dfrac{\langle x,y\rangle}{\|x\|\cdot\|y\|} \rightsquigarrow -1 \leq \dfrac{\langle x,y\rangle}{\|x\|\cdot\|y\|} \leq 1 \rightsquigarrow x \perp y \Leftrightarrow \langle x,y\rangle = 0$
- $\langle x,y\rangle = \dfrac{\|x+y\|^2 - \|x\|^2 - \|y\|^2}{2}$
- Parallelogram Law: $\exists \langle\cdot,\cdot\rangle$ inducing $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle} \Leftrightarrow \|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad \forall x,y \in X$ 

- $a,b \geq 0 : p,q > 1$ ST $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$
- Holder: $\begin{cases} x,y \in \mathbb{R}^n : p,q > 1 \text{ ST } \frac{1}{p}+\frac{1}{q} = 1 \Rightarrow \sum_{k=1}^n |x_k|\cdot|y_k| \leq \left(\sum_{k=1}^n |x_k|^p\right)^{1/p} \cdot \left(\sum_{k=1}^n |y_k|^q\right)^{1/q} \\ f,g \in C[a,b] : \underline{\qquad\qquad} \Rightarrow \int_a^b |f(x)|\cdot|g(x)|\,dx \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} \cdot \left(\int_a^b |g(x)|^q dx\right)^{1/q} \end{cases}$
- Minkovski: $\begin{cases} x,y \in \mathbb{R}^n : p \geq 1 \Rightarrow \left(\sum_{k=1}^n (|x_k|+|y_k|)^p\right)^{1/p} \leq \left(\sum_{k=1}^n |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^n |y_k|^p\right)^{1/p} \\ f,g \in C[a,b] : p \geq 1 \Rightarrow \left(\int_a^b |f(x)+g(x)|^p dx\right)^{1/p} \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} + \left(\int_a^b |g(x)|^p dx\right)^{1/p} \end{cases}$
  $\Rightarrow \|fg\|_1 \leq \|f\|_p \|g\|_q$
  $\Rightarrow \|(|f|+|g|)\|_p \leq \|f\|_p + \|g\|_p$
- ℓ^p spaces (fin dim) : $X = \mathbb{R}^n : \|x\|_p = \left(\sum_{k=1}^n |x_k|^p\right)^{1/p} \quad \forall p > 1 \; ; \; \|x\|_\infty = \max_{1 \leq k \leq x} |x_k| = \lim_{p\to\infty} \|x\|_p \; : \; \|\cdot\|_p \Leftrightarrow \|\cdot\|_q \Leftrightarrow \|\cdot\|_\infty$
- L^p spaces (∞ dim) : $X = C[a,b] : \|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{1/p} \quad \forall p > 1 \; ; \; \|f\|_\infty = \sup_{x\in[a,b]} |f(x)| = \lim_{p\to\infty} \|f\|_p \; : \; \|\cdot\|_p \not\Leftrightarrow \|\cdot\|_q$ for $p \neq q$; $\|\cdot\|_p \not\Leftrightarrow \|\cdot\|_\infty$
- Equivalence: $\|\cdot\|_1 \sim \|\cdot\|_2 \Leftrightarrow \exists\, c_1, c_2 > 0$ ST : $c_1\|x\|_2 \leq \|x\|_1 \leq c_2\|x\|_2 \quad \forall x \in X \longrightarrow$ prop: $\dim X < \infty \Rightarrow \|\cdot\|_1 \sim \|\cdot\|_2$ always
- Pseudo metrics: $x = y \Rightarrow \rho(x,y) = 0 \longrightarrow$ prop: $(X/\sim, d)$ is a metric space: $\begin{cases} x \sim y \Leftrightarrow \rho(x,y) = 0 \\ d([x],[y]) = \rho(x,y) \end{cases} \quad \forall x,y \in X$
- Product spaces:
  - finite: $\prod_{k=1}^n X_k = \{x = (x_1,\dots,x_n) : x_k \in X_k\} \; ; \; d(x,y) = \left(\sum_{k=1}^n (d_k(x_k,y_k))^2\right)^{1/2}$
  - countable: $\prod_{k=1}^\infty X_k = \{x = (x_k)_{k=1}^\infty : x_k \in X_k\} \; ; \; d(x,y) = \sum_{k=1}^\infty 2^{-k} \cdot \dfrac{d_k(x_k,y_k)}{1+d_k(x_k,y_k)}$

Open, closed sets & Convergence:

- open ball: $B(x,\epsilon) = \{y \in X : d(x,y) < \epsilon\}$ → Prop: open ball = open set
- open set: $O \subset X$: $\forall x \in O, \exists r_x > 0$ st $B(x, r_x) \subset O$ → Ex: X and ϕ are always open
- Neighborhood of x: open set containing x.
- $O \subset X$ open $\Leftrightarrow O = \bigcup_{i \in I} B(x_i, r_i)$
- $Y \subset X$: $O_Y \subset Y$ open $\Leftrightarrow O_Y = O_X \cap Y$: $O_X \subset O$ open
- $A \subset X$: $x \in X$ is a closure pt of $A \Leftrightarrow \forall O \ni x : O \cap A \neq \phi$
- $A \underset{\subset X}{}$ closed $\Leftrightarrow A = cl(A)$ → prop: $cl(A)$ = smallest closed set containing A. → Ex: X & ϕ are closed too!
  $\Leftrightarrow A^c$ is open in X $\quad \Rightarrow cl(A) = \bigcap F$ for $F \supset A$, F closed

Prop: $\bigcup_{i \in I}$ open = open ; $\bigcap_{i \in I}$ closed = closed ; $\bigcup_{i=1}^{N}$ closed = closed ; $\bigcap_{i=1}^{N}$ open = open $\quad (N < \infty)$

- $Int(A) = \bigcup_{O \subset A} O$ = largest open set contained in A.
- $\partial A = cl(A) \cap cl(A^c)$ $\Rightarrow \partial A$ is closed → prop: $\partial \mathbb{Q} = \mathbb{R} = \partial(\mathbb{Q}^c)$; $\partial \mathbb{R} = \phi$

ID: $A \subset B \Rightarrow cl(A) \subset cl(B)$; $A, B \subset X \Rightarrow cl(A \cup B) = cl(A) \cup cl(B)$

$\partial A \cap int(A) = \phi$; $cl(A) = int(A) \cup \partial A$; $A, B \subset X \Rightarrow \partial(A \cup B) \subset (\partial A) \cup (\partial B)$

Convergence, Continuity etc:

unique if it exists

- $(x_n)_{n=1}^{\infty} \to x \in (X,d) \iff \lim_{n\to\infty} d(x_n,x)=0 \iff \forall \varepsilon>0, \exists N\in\mathbb{N}$ ST $d(x_n,x)<\varepsilon \ \forall n\geq N$.
- $(x_n)_{n=1}^{\infty}$ bounded $\iff \exists y\in X, M>0$ ST $d(x_n,y)\leq M \ \forall n\geq 1 \iff (d(x_n,y))_{n=1}^{\infty}$ is bounded in $\mathbb{R}$.
- $d_1 \sim d_2$ and $x_n \to x\in (X,d_1) \implies x_n \to x \in (X,d_2)$
- $X=\prod_{k=1}^{\infty} X_k = \{x=(x_k)_{k=1}^{\infty} : x_k\in X_k\}$: $(x^{(n)})_{n=1}^{\infty} = ((x_k^{(n)})_{k=1}^{\infty})_{n=1}^{\infty}$, $x=(x_k)_{k=1}^{\infty}$ → prop: $d(x^{(n)},x)\to 0 \iff d(x_k^{(n)},x_k)\to 0 \ \forall k\in\mathbb{N}$

Prop: (x_n) : Convergent $\implies$ Bounded

$\boxed{\text{Prop:}}$ $A\subset X$: $x\in cl(A) \iff \exists (x_n)_{n=1}^{\infty}\in A$ ST $\lim_{n\to\infty} x_n = x$ $\implies$ A is closed $\iff$ ("$(x_n)\in A$ converges to x" $\implies$ "$x\in A$")

- $d_1\sim d_2$: A open in $(X,d_1) \iff$ A open in (X,d_2) ; A closed in $(X,d_1)\iff$ A closed in (X,d_2)

continuity: $f:(X,d_X)\to (Y,d_Y)$ cont on X : → $\forall x\in X$: $(\forall\varepsilon>0)(\exists\delta>0)(f(B_X(x,\delta))\subset B_Y(f(x),\varepsilon))$

- $\forall x\in X$: $\forall (x_n)_{n=1}^{\infty}\in X$, $\lim_{n\to\infty} d_X(x_n,x)=0 \implies \lim_{n\to\infty} d_Y(f(x_n),f(x))=0$
- $\forall x\in X$: $(\forall\varepsilon>0)(\exists\delta_{(x,\varepsilon)}>0)(d_X(x,y)<\delta \implies d_Y(f(x),f(y))<\varepsilon)$, $\forall y\in X$
- $\forall O_Y\subset Y$ open : $f^{-1}(O_Y)=O_X\subset X$ open

Comp: $f:(X,d_X)\to(Y,d_Y)$, $g:(Y,d_Y)\to(Z,d_Z)$ cont $\implies g\circ f:(X,d_X)\to(Z,d_Z)$ cont

unif cont: $(\forall\varepsilon>0)(\exists\delta_{(\varepsilon)}>0)(\forall x,y\in X)(d_X(x,y)<\delta \implies d_Y(f(x),f(y))<\varepsilon)$

Lipschitz: $(\exists c>0)(d_Y(f(x),f(y))\leq c\cdot d_X(x,y), \forall x,y\in X)$

Prop: Lipschitz $\implies$ unif cont $\implies$ Cont

X Compact: f Cont on X $\implies f$ unif cont on X

Complete Metric Spaces:

- Cauchy Seq: $(x_n)_{n=1}^{\infty} \in X$: $\forall \varepsilon > 0, \exists N > 0$ ST $d(x_n, x_m) < \varepsilon \ \forall n, m \geq N$.
- Prop: • x_n Converges$^{(\text{in } X)}$ $\Rightarrow$ Cauchy. ; $\begin{cases} (x_n) \text{ Cauchy} + \text{Conv Subseq} \Rightarrow (x_n) \text{ Conv} \\ (x_n) \text{ Cauchy} + \underline{\text{Not}} \text{ Conv} \Rightarrow \text{No Conv Subseq} \end{cases}$
  • X complete: converges (in X) $\Leftrightarrow$ Cauchy.

- **X complete**: $\forall (x_n)_{n=1}^{\infty} \in X$ Cauchy $\Rightarrow$ Converge in X. WRT d.

How To SHOW X is complete: $\begin{cases} \bullet (x_n) \text{ Cauchy in } (X, d) \Rightarrow (x_n) \text{ Cauchy in } (\mathbb{R}, |\cdot|) \\ \bullet (\mathbb{R}, |\cdot|) \text{ Complete} \Rightarrow x_n \to x \\ \bullet x \in X \\ \bullet x_n \to x \text{ in } (X, d) \end{cases}$

Cantor intersect° thm: (X, d) complete $\Leftrightarrow \forall (F_n)$ contracting seq of nonempty closed subsets of X : $\exists x \in X$ ST $\bigcap_{n=1}^{\infty} F_n = \{x\}$

- Contracting Seq: descending Seq of nonempty subsets of (X, d) : $Y_1 \supset Y_2 \supset \cdots \supset Y_n \supset \cdots$ ST $\lim_{n \to \infty} \operatorname{diam}(Y_n) = 0$
- $\phi \neq (Y, d) \subset (X, d)$: $\operatorname{diam}(Y) = \sup\{d(x,y) : x, y \in Y\}$ $\to$ Y bounded $\Leftrightarrow$ diam Y $< \infty$

Prop: $\operatorname{diam} Y = \operatorname{diam}(cl(Y))$

- $(Y, d) \subset \overset{\to \text{complete}}{(X, d)}$: Y complete $\Leftrightarrow$ Y Closed

- $(X, \|\cdot\|)$ a normed space: complete $\Rightarrow$ Banach space $\xRightarrow{\text{if } \|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}}$ Hilbert space

- Extension by unif cont: $S \subset X$ dense, Y complete : $f: S \to Y$ unif cont $\Rightarrow \exists! \tilde{f}$ ST: $\begin{cases} \tilde{f}: X \to Y \\ \tilde{f}|_S = f. \\ \tilde{f} \text{ unif cont} \end{cases}$
- Dense: $S \subset X$ dense $\Leftrightarrow cl(S) = X \to \forall x \in X : \exists (x_n) \in S$ ST $x_n \to x$
- Isometry: $(X, d_X) \cong (Y, d_Y)$: $\exists$ a bij° $f: X \leftrightarrow Y$ ST $d_X(x,y) = d_Y(f(x), f(y)) \ \forall x, y \in X$
- Completion of a MS: (Y, d_Y) is a complet° of (X, d_X) if $\exists \tilde{Y} \subset Y$ ST: $\begin{cases} \bullet\ cl(\tilde{Y}) = Y \\ \bullet\ Y \text{ is complete} \\ \bullet\ \exists \text{ a bij° } f: X \leftrightarrow \tilde{Y} \text{ ST } d_X(x,y) = d_Y(f(x), f(y)) \end{cases}$

thm: Any metric Space has a complet° (unique up to isometry) $\Rightarrow$ Normed Space $\to$ Banach Space $\xrightarrow[\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}]{}$ Hilbert Space

- $X = \prod_{k=1}^{N} X_k$ (Note: $N \leq \infty$) : (X, d) complete $\Leftrightarrow (X_k, d_k)$ complete $\forall 1 \leq k \leq N$ $\begin{cases} d(x,y) = \sum_{k=1}^{\infty} 2^{-k} \dfrac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)} \\ d(x,y) = \left(\sum_{k=1}^{N} (d_k(x_k, y_k))^2 \right)^{1/2} \end{cases}$

Ex: • $(C([a,b]), \|\cdot\|_p)$: $1 \leq p < \infty \to$ NOT complete! ; $p = \infty \to$ Banach.
- $\ell^p(\mathbb{N})$: Banach $\forall 1 \leq p \leq \infty$ (Note: $\ell^\infty(\mathbb{N}) = \{(x_n)_{n=1}^{\infty} : \|x\|_\infty = \sup |x_n| < \infty\}$)
- ℓ^p : Complete $\forall 1 \leq p \leq \infty$ $\to$ But not Separable if $p = \infty$
- $(C(X), \|\cdot\|_\infty)$ Banach for X compact

Compact metric spaces:

- <u>Cover</u>: $\begin{cases} X: \{O_\alpha\}_{\alpha\in I} \text{ covers } X \text{ if } X = \bigcup_{\alpha\in I} O_\alpha \\ Y \subset X: \{O_\alpha\}_{\alpha\in I} \text{ covers } Y \text{ if } Y \subset \bigcup_{\alpha\in I} O_\alpha \end{cases}$ $\longrightarrow$ open cover: O_α open ; finite cover: $|I| < \infty$
- Finite intersect° prop: $\{F_\alpha\}_{\alpha\in I}$ ST $\forall$ finite subfamily $F_{\alpha_1}, \dots, F_{\alpha_n}$: $\bigcap_{i=1}^n F_{\alpha_i} \neq \phi$.
- Totally bounded: • $\forall \varepsilon > 0$, $\exists x_1, \dots, x_n \in X$ ST $X = \bigcup_{i=1}^n B(x_i, \varepsilon)$ $\Rightarrow$ "Totally bounded $\Rightarrow$ Bounded"
  • $Y \subset X$: $\forall \varepsilon > 0$, $\exists y_1, \dots, y_n \in Y$ ST $Y \subset \bigcup_{i=1}^n B(y_i, \varepsilon)$

  <u>Prop</u>: • $(X, \|\cdot\|)$: $Cl(B(0,1)) = \{x \in X : \|x\| \leq 1\}$ Totally bounded $\Rightarrow$ $\dim X < \infty$.
  • $(X, \|\cdot\|)$, $\dim X < \infty$, $Y \subset X$: Y bounded $\iff$ Totally bounded.

- <u>Compactness</u> = X compact : every open cover of X in X has a finite subcover
  $\iff$ X complete + totally bounded
  $\iff$ X Sequentially compact $\rightarrow \forall$ Seq $(x_n)_{n=1}^\infty \in X$: $\exists (x_{n_k})_{k=1}^\infty \in X$ ST $x_{n_k} \longrightarrow x \in X$.
  $\iff$ $\forall \{F_\alpha\}_{\alpha\in I}$, $F_\alpha \subset X$ closed: if $\{F_\alpha\}_{\alpha\in I}$ has finite intersect° prop $\Rightarrow \bigcap_{\alpha\in I} F_\alpha \neq \phi$
- X has finitely many pts $\Rightarrow$ X is compact.
- $Y \subset X$: • X compact: Y compact $\iff$ Y closed.
  • Y compact $\Rightarrow$ Y closed & bounded
  • Y compact in Y $\iff$ Every open cover of Y in X has a fin subcover $(V_\alpha = O_\alpha \cap Y)$
- $(X, \|\cdot\|)$, $Y \subset X$: $\dim X < \infty$: Y compact in $(X, \|\cdot\|)$ $\iff$ Y closed & bounded in $(X, \|\cdot\|)$ (Heine-Borel)
  $\hookrightarrow$ any 2 norms $\|\cdot\|_1 \sim \|\cdot\|_2$ are equiv
- $(X, \|\cdot\|)$: $\dim X = \infty$ $\Rightarrow$ $B = \{x \in X : \|x\| = 1\} \subset (X, \|\cdot\|)$ is <u>NOT</u> compact. $\longrightarrow$ closed unit ball compact $\iff \dim X < \infty$
  • Reisz's lemma: $(X, \|\cdot\|)$, $S \subsetneq X$, S closed : $(\forall\, 0 < \alpha < 1)(\exists y_\alpha \in X \setminus S)(\forall x \in S)(\|y_\alpha - x\| > \alpha$ and $\|y_\alpha\| = 1)$
  • Prop: $(X, \|\cdot\|)$, $S \subset X$: $\dim S < \infty \Rightarrow S$ closed
- $f : X \to Y$ cont: X compact $\Rightarrow$ $\begin{cases} f(X) \subset Y \text{ compact.} \\ f \text{ achieves min/max on } X. \\ f \text{ unif cont on } X \end{cases}$

- <u>Lebesgue nb for cover $\{O_\alpha\}_{\alpha\in I}$</u> = X compact, $\{O_\alpha\}_{\alpha\in I}$ open cover of X $\Rightarrow (\exists\, \varepsilon > 0)(\forall x \in X)(\exists \alpha \in I)(B(x, \varepsilon) \subset O_\alpha)$
  (ε: indep of x)

- $X = \prod_{i=1}^{N} X_k$ ($N \leq \infty$) : (X_k, d_k) compact $\forall 1 \leq k \leq N \Rightarrow (X, d)$ is compact.

- <u>Ex</u>: <u>Compact</u>: • $B = \{x \in X : \|x\| = 1\} \subset (X, \|\cdot\|)$ ($\dim X < \infty$)
  • $Y \subset X$: X compact & Y closed
  • $Y \subset X$: $\dim X < \infty$, Y closed & bounded
  • $|X| < \infty$ (finitely many pts)

  <u>Not Compact</u>: • $B = \{x \in X : \|x\| = 1\} \subset (X, \|\cdot\|)$ ($\dim X = \infty$)
  • $(C([a,b]), \|\cdot\|_p)$

Separable Metric Spaces:

- Countable: $\exists\, f: \mathbb{N} \to S$, $f(k) = s_k$ $\begin{cases} \text{Surjective if } |S| < \infty \\ \text{Bijective if } |S| = \infty \end{cases}$ $\Rightarrow S = \{s_1, s_2, \dots, s_n, \dots\}$

  Prop: $S' \subset S$: S Countable $\Rightarrow S'$ Countable ; $\bigcup_{i=1}^{\infty}$ Countable = Countable.

  $N < \infty$: $\prod_{i=1}^{N}$ Countable = Countable

  Ex: • $\mathbb{N}, \mathbb{Q}, \mathbb{Z}$ are countable. But $[0,1]$, $[a,b]$, $\mathbb{R}$ are uncountable.

  • $S = \{(s_n)_{n=1}^{\infty} : s_n \in \{0,1\}\}$ is uncountable

- Dense: $S \subset X$ is dense $\Leftrightarrow X = cl(S)$ $\quad (\forall x \in X)(\exists (x_n)_{n=1}^{\infty} \in S)(\lim_{n\to\infty} x_n = x \in X)$

  $\Leftrightarrow \forall Y \subset X$ open & nonempty : $Y \cap S \neq \phi$

- Separability: X is separable $\Leftrightarrow \exists\, S \subset X$ ST S countable & dense $\quad cl(S) = X$

  $\Leftrightarrow \exists$ a countable collect^n $\{O_n\}_{n=1}^{\infty}$ of nonempty open sets in X ST any open set in X can be written as a union of subcoll^n of $\{O_n\}_{n=1}^{\infty}$

- Prop: X separable $\Rightarrow \forall$ family $\{B(x_i, r_i)\}_{i\in I}$ of disjoint open balls in X : I is Countable.

  Corr: $\exists$ an uncountable family of disjoint open balls in X $\{B(x_i, r_i)\}_{i\in I} \Rightarrow X$ is NOT separable!

- Prop: • X compact $\Rightarrow X$ separable.

  • $(X, \|\cdot\|)$: $\dim X < \infty \Rightarrow X$ separable

  • $\{(X_k, d_k)\}_{k=1}^{N \text{ or } \infty}$ separable $\Rightarrow (X, d)$ separable $\quad (X = \prod_{k=1}^{N \text{ or } \infty} X_k)$

  • X separable $\Rightarrow Y \subset X$ separable

- Bernstein: • $f \in (C([0,1]), \|\cdot\|_\infty)$: $\forall \varepsilon > 0$, $\exists\, P_n(x)$: $\|f - P\|_\infty = \max_{0\le x\le 1} |f(x) - P(x)| < \varepsilon$ $\forall n \geq n(\varepsilon)$ $\quad \exists\, n(\varepsilon) \geq 0$ ST

  $P_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \cdot \binom{n}{k} x^k (1-x)^{n-k}$

  • f Lipschitz on $[0,1]$ $\Rightarrow n(\varepsilon) = \dfrac{4K^2 \|f\|_\infty}{\varepsilon^3}$
  $\hookrightarrow |f(x) - f(y)| \leq K|x-y|$

  • $(C([0,1]), \|\cdot\|_\infty)$ is separable $\longrightarrow S = \bigcup_{N=0}^{\infty} S_N$, $S_N = \{r_0 + r_1 x + \dots + r_N x^N : r_k \in \mathbb{Q}\}$

  • $(C([a,b]), \|\cdot\|_\infty)$ is separable

- Ex:

Separable:	Not Separable
$(\mathbb{R}, \lvert\cdot\rvert)$, $(\mathbb{R}^n, \lvert\cdot\rvert)$	$(C_b(\mathbb{R}), \|\cdot\|_\infty)$: space of all bounded cont f^n: $f: \mathbb{R} \to \mathbb{R}$
$1 \leq p \leq \infty$: $(C([0,1]), \|\cdot\|_p)$ & $(C([a,b]), \|\cdot\|_p)$	
X Compact,	
$(X, \|\cdot\|)$: $\dim X < \infty$	
$\prod_{i=1}^{\infty}$ (separable)	
$Y \subset$ Separable	

Arzelà - Ascoli theorem :

Recall : • X compact $\Rightarrow (C(X), \|\cdot\|_\infty)$ is Banach.

• $\dim Y < \infty$: $S \subset Y$ compact $\Leftrightarrow$ S closed & bounded in $(Y, \|\cdot\|)$

(AA) • $\dim(C(X)) = \infty$: $D \subset C(X)$ compact $\Leftrightarrow$ D closed + bounded + equicontinuous in $(C(X), \|\cdot\|)$

Equicontinuity : $D \subset C(X)$ equicont on X : $(\forall \varepsilon > 0)(\exists \underbrace{\delta > 0}_{\text{indep of } f})(\forall f \in D)(\forall x, y \in X)\,(d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon)$

$\neq$ unif cont : $(\forall \varepsilon > 0)(\forall f \in D)(\exists \delta_f > 0)(\forall x, y \in X)(d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon)$

• AA : X compact, $D \subset C(X)$: D compact $\Longleftrightarrow$ D closed, bounded & equicont.
$(X, \|\cdot\|)$ in $(C(X), \|\cdot\|)$

Corr : X compact, $D \subset C(X)$: $cl(D)$ compact $\Leftrightarrow$ D bounded & equicont
in $(C(X), \|\cdot\|)$

• thm : X separable, $(f_n)_{n=1}^\infty$ a seq of funct$^{\circ}$: (f_n) bounded + equicont $\Rightarrow \exists (f_{n_k}) \longrightarrow g$, g unif cont on X
$f_n : X \to \mathbb{R}$ $\quad \exists M > 0 : \sup_{x \in X} |f(x)| \leq M$ $\quad \lim_{k \to \infty} f_{n_k}(x) = g(x)\ \forall x \in X$

• EX : $X = [0,1]$, $K_1, K_2 > 0$ constants : $D \subset C([0,1])$ a collect$^{\circ}$ of f° ST :

• $\begin{cases} \sup_{x \in X} |f(x)| \leq K_1 \quad \forall f \in D \\ \sup_{x \in X} |f'(x)| \leq K_2 \quad \forall f \in D \end{cases} \Rightarrow cl(D)$ compact in $(C([0,1]), \|\cdot\|)$

↳ every f is diff on $(0,1)$

• $\begin{cases} |f(0)| \leq K_1 & \forall f \in D \\ \int_0^1 |f'(x)|^2 dx \leq K_2 & \forall f \in D \\ f \text{ continuously diff on } [0,1] & \forall f \in D \end{cases} \Rightarrow cl(D)$ compact in $(C([0,1]), \|\cdot\|)$

↳ f' $\exists$ on $(0,1)$, f' cont on $[0,1]$ by extension

Baire Category theorem:

BCT: (X,d) Complete: $\{O_n\}_{n=1}^{\infty}$ a countable collect° of open & dense sets in $X \Rightarrow \bigcap_{n=1}^{\infty} O_n$ is dense in X.

i.e. $\forall O \subset X$ nonempty: $\left(\bigcap_{n=1}^{\infty} O_n\right) \cap O \neq \phi$

Corr: (X,d) complete: $\{F_n\}_{n=1}^{\infty}$ a countable collect° of closed sets in X;

if $Int\left(\bigcup_{n=1}^{\infty} F_n\right) \neq \phi$ then $Int(F_n) \neq \phi$ for at least one $n \in \mathbb{N}$.

Nowhere dense set: $A \subset X$ ST $Int(Cl(A)) = \phi$

First category set: $A \subset X$ ST $A = \bigcup_{i=1}^{\infty} B_i$, B_i nowhere dense.

2nd category set: $A \subset X$ ST A is NOT of 1st category.

BCT: Any complete metric space is of 2nd category

G_δ set: $A \subset X$ ST $A = \bigcap_{n=1}^{\infty} O_n$, O_n open in X

Ex: any point $x \in X$ is a G_δ set: $\{x\} = \bigcap_{n=1}^{\infty} B(x, \frac{1}{n})$

F_δ set: $A \subset X$ ST $A = \bigcup_{n=1}^{\infty} F_n$, F_n closed in X

Prop: (X,d) complete: $\{V_n\}_{n=1}^{\infty}$ a countable coll° of dense G_δ sets $\Rightarrow \bigcap_{n=1}^{\infty} V_n$ is dense + G_δ

Isolated pt: (X,d): $x \in X$ isolated if $\exists \epsilon > 0$ ST $B(x,\epsilon) \cap X = \{x\}$

Prop: (X,d) complete: $S \subset X$ countable & dense $\Rightarrow S$ is not G_δ

Corr: $\{O_n\}_{n=1}^{\infty}$ open & dense in $X \Rightarrow \bigcap_{n=1}^{\infty} O_n$ is dense + G_δ ⊕ uncountable

if X has No isolated pts

Banach Fixed point / Contraction principle :

Contraction : (X,d) : $f: X \longrightarrow X$ is a contraction : $\exists \alpha \in (0,1)$ ST : $d(f(x),f(y)) \leq \alpha\, d(x,y)$ $\forall x,y \in X$

Prop : f is a contraction $\Longrightarrow$ f Lipschitz $\longrightarrow$ f unif cont

BFP : (X,d) complete : $f: X \longrightarrow X$ a contraction $\Longrightarrow$ $\exists!\, x \in X$ ST $f(x) = x$

Rate of convergence : $d(x, x_N) \leq \dfrac{\alpha^N}{1-\alpha}\, d(x_1, x_0)$ where $x_n := f(x_{n-1})$

Generalizat° :
- (X,d) compact : $f: X \to X$ ST $d(f(x),f(y)) < d(x,y)$ $\forall x \neq y$ $\Longrightarrow \exists!\, x \in X$ ST $f(x) = x$
- (X,d) complete, $f: X \to X$: if $\exists!\, x \in X$ ST $f^k(x) = x$ ($f^k = f \circ f \circ \dots \circ f$ k times) for some $k \in \mathbb{N}$, then $\exists!\, x \in X$ ST $f(x) = x$

  Corr : if f^k is a contract° for some $k \in \mathbb{N}$, then $\exists!\, x \in X$ ST $f(x) = x$.
- $(X, \|\cdot\|_\infty)$ Banach, $A: X \to X$ a cont. linear map :
  if $A^k = A \circ A \circ \dots \circ A$ is a contract° for some $k \in \mathbb{N}$, then $\exists!\, x \in X$ ST : $x = A(x) + y$ $\forall y \in X$

Integral Equations :
- $f(x) = \lambda \int_a^b K(x,y) f(y)\, dy + g(x)$ $\quad \forall x \in [a,b]$ (*) (every f° is cont on $[a,b]$ or $[a,b] \times [a,b]$)
- $f(x) = \lambda \int_a^x K(x,y) f(y)\, dy + g(x)$ $\quad \forall x \in [a,b]$ (**) ( )

Compact linear map : $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$: $A : X \longrightarrow Y$ a continuous linear map :

A is compact if $cl\{A(f) : \|f\|_X \leq 1\} \subset Y$ is compact in $(Y, \|\cdot\|_Y)$

Ex : $A: C([a,b]) \longrightarrow C([a,b])$, $A(f) = \int_a^b K(x,y) f(y)\, dy$ is a compact operator (A-A thm)

(*) has a unique sol° when $0 < |\lambda| \cdot M(b-a) < 1$

Particular case : $f(x) = \lambda \int_0^1 f(y)\, dy + g(x)$: $\exists!$ sol if $\lambda \neq 1$, and if $\lambda = 1$: $\begin{cases} \text{if } g = 0 \to \infty \text{ of solutions} \\ \text{if } g \neq 0 \to \emptyset \text{ solut°} \end{cases}$

(**) has a unique solut° $\forall \lambda \in \mathbb{R}$.