Part 1:

Abstract Integration

Measurability :

(I) Measurable Spaces :

σ-Algebra : X a set, $\mathcal{F}$ a family of subsets of X :

→ $\mathcal{F}$ is a σ-Algebra if : $\begin{cases} \bullet\ X \in \mathcal{F} \\ \bullet\ A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F} \\ \bullet\ (A_n)_{n=1}^{\infty} \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F} \end{cases}$

→ $(X, \mathcal{F})$ is a measurable space

→ Elements of $\mathcal{F}$ are measurable sets

Note : $\bigcap_{n=1}^{\infty} A_n = \Big(\underbrace{\bigcup_{n=1}^{\infty} A_n^c}_{\in \mathcal{F}}\Big)^c \in \mathcal{F}$

Prop :
- $\phi \in \mathcal{F}$
- $(A_n)_{n=1}^{\infty} \in \mathcal{F} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$
- $A, B \in \mathcal{F} \Rightarrow A \setminus B \in \mathcal{F}$

(Pf) $A \setminus B = A \cap B^c$

Ex :
- Power set of X : $\mathcal{P}(X)$ = collection of all subsets of X, $\mathcal{P}(X)$ is a σ-Alg
- Trivial σ-Alg in X : $\mathcal{F} = \{\phi, X\}$

σ-Alg Generated by a set A : X a set, A a family of subsets of X.

Then $\mathcal{F}_A = \bigcap_{\mathcal{F} \in \mathcal{R}} \mathcal{F}$, $\mathcal{R} = \{\mathcal{F} : \mathcal{F}$ is a σ-Alg in X, $A \subset \mathcal{F}\}$

⇨ $\mathcal{F}_A$ is the <u>smallest</u> σ-Alg containing A

(Pf)
- $\mathcal{P}(X) \in \mathcal{R} \Rightarrow \mathcal{R} \neq \phi$.
- $X \in \mathcal{F}\ \forall \mathcal{F} \Rightarrow X \in \bigcap_{\mathcal{F} \in \mathcal{R}} \mathcal{F} = \mathcal{F}_A$
- $A \in \mathcal{F}_A \Rightarrow A \in \mathcal{F}\ \forall \mathcal{F} \Rightarrow A^c \in \mathcal{F}\ \forall \mathcal{F}$
  $\Rightarrow A^c \in \mathcal{F}_A = \bigcap_{\mathcal{F} \in \mathcal{R}} \mathcal{F}$
- $(A_n)_{n=1}^{\infty} \in \mathcal{F}_A \Rightarrow (A_n)_{n=1}^{\infty} \in \mathcal{F}\ \forall \mathcal{F}$
  $\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}\ \forall \mathcal{F}$
  $\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}_A = \bigcap_{\mathcal{F} \in \mathcal{R}} \mathcal{F}$

(II) Measurable functions :

Measurable function : $(X, \mathcal{F})$ a meas. space, (Y, τ) a top space

$f : X \longrightarrow Y$ is measurable $\Leftrightarrow \forall V \in \tau$ open : $f^{-1}(V) \in \mathcal{F}$

⚠ $f, g : X \to Y$ measurable $\not\Rightarrow$ $f \circ g$ measurable

Prop : $(X, \mathcal{F})$ a MSp, (Y, τ_Y) and (Z, τ_Z) Top Spaces : $\begin{cases} f : X \longrightarrow Y \text{ measurable} \\ g : Y \to Z \text{ continuous} \end{cases}$

Then $h = g \circ f : X \longrightarrow Z$ is a meas. funct°

(Pf) $X \xrightarrow{f} Y \xrightarrow{g} Z$
$f^{-1}(g^{-1}(V))$ $g^{-1}(V)$ V

$V \in \tau_Z$ open :

So $g^{-1}(V) \in \tau_Y$ (g cont)

So $f^{-1}(g^{-1}(V)) \in \mathcal{F}$ (f meas)

$\Rightarrow h^{-1}(V) \in \mathcal{F}$.

Prop : $\begin{cases} (X, \mathcal{F}) \text{ a MSp} : u, v : X \longrightarrow \mathbb{R} \text{ meas } f° \\ (Y, \tau) \text{ a Top Space} : \Phi : \mathbb{R}^2 \to Y \text{ cont } f° \end{cases} \Rightarrow \begin{cases} f : X \longrightarrow Y \\ f(x) = \Phi(u(x), v(x)) \end{cases}$ is <u>measurable</u>.

Proof : Let $g(x) = (u(x), v(x))$: $g : X \to \mathbb{R}^2$ $\Rightarrow f = \Phi \circ g$: Φ cont $\Rightarrow$ "f meas $\Leftrightarrow$ g meas"

- Let $V = (a,b) \times (c,d) \subset \mathbb{R}^2$ open : $g^{-1}(V) = \{x \in X : g(x) \in (a,b) \times (c,d)\} = \{x \in X : (u(x), v(x)) \in (a,b) \times (c,d)\}$
  $= \{x \in X : u(x) \in (a,b), v(x) \in (c,d)\}$
  $= \underbrace{u^{-1}((a,b))}_{\in \mathcal{F}} \cap \underbrace{v^{-1}((c,d))}_{\in \mathcal{F}} \in \mathcal{F}$ (Pf) u,v <u>meas</u> !
- $V \subset \mathbb{R}^2$ an arbitrary open set : Then $V = \bigcup_{n=1}^{\infty} V_n$, $V_n = (a_n, b_n) \times (c_n, d_n)$
  So $g^{-1}(V) = \bigcup_{n=1}^{\infty} g^{-1}(V_n) \in \mathcal{F}$ because $g^{-1}(V_n) \in \mathcal{F}\ \forall n$

Corr : X a meas Space :
- $f, g : X \longrightarrow \mathbb{R}$ measurable $f°$ $\Rightarrow$ $f+g$, $f \cdot g$ are meas $f°$
- $f : X \longrightarrow \mathbb{C}$, $f(x) = u(x) + i v(x)$ with $u, v : X \to \mathbb{R}$: f meas $\Leftrightarrow$ u, v meas

  $\boxed{\Rightarrow}$ f meas : Take $g_1 : \mathbb{C} \to \mathbb{R}$, $g_1(a+ib) = a$
  g_1 cont, $g_1 \circ f = u \Rightarrow u$ is meas (Similar for V)

  $\boxed{\Leftarrow}$ $\Phi : \mathbb{R}^2 \to \mathbb{C}$, $\Phi(a,b) = a+ib$ is an ISOMETRY $\Rightarrow f = \underset{= u+iv}{\Phi(u,v)}$
  So u, v meas $\Rightarrow$ f meas

Note : f meas $\Rightarrow |f| = \sqrt{u^2+v^2}$ is measurable

(Pf) $\begin{cases} u, v \text{ meas } f° \\ \sqrt{u^2+v^2} \text{ is continuous} \end{cases}$

characteristic function: $(X, \mathcal{F})$ a meas. space, $A \subset X$ a set:

$$\chi_A : X \to \mathbb{R}, \quad \chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases} \implies \chi_A^{-1}(V) = \begin{cases} \phi, & \text{if } 1 \notin V, 0 \notin V \\ A, & \text{if } 1 \in V, 0 \notin V \\ A^c, & \text{if } 1 \notin V, 0 \in V \\ X, & \text{if } 1 \in V, 0 \in V \end{cases}$$

Prop = χ_A measurable $\iff A \in \mathcal{F}$

Prop = $(X, \mathcal{F})$ a MSp, $f : X \to \mathbb{C}$ a MF° : Then $\exists$ a MF° $\alpha : X \to \mathbb{C}$

Then $\exists$ a MF° $\alpha : X \to \mathbb{C}$ ST: $\begin{cases} |\alpha(x)| = 1, & \forall x \in X \\ \text{and } f(x) = \alpha(x) \cdot |f(x)|, & \forall x \in X \end{cases}$ $\boxed{\alpha(x) = \dfrac{f(x) + \chi_E(x)}{|f(x) + \chi_E(x)|}}$

Proof = note that $\alpha(x) = \dfrac{f(x)}{|f(x)|}$ fails when $f(x) = 0$

- Let $E = \{x \in X : f(x) = 0\} \implies E$ is a MSet
- $E^c = \{x \in X : f(x) \neq 0\} = f^{-1}(\mathbb{C} \setminus \{0\}) \implies Y := \mathbb{C} \setminus \{0\}$: let $\begin{cases} \Phi : Y \to \mathbb{C} \\ \Phi(z) = \dfrac{z}{|z|} \end{cases} \implies \Phi$ is continuous on Y.
- Let $h(x) = f(x) + \chi_E(x)$ $\forall x \in X$: $h(x) \neq 0$ $\forall x \in X$ and h is a MF°.
- Let $\alpha(x) = \Phi(h(x))$ $\forall x \in X$: Φ cont, h meas $\implies \alpha$ is a MF° ! (??) $\Phi^{-1}(V)$ open in $\mathbb{C} \setminus \{0\} \implies \Phi^{-1}(V)$ open in $\mathbb{C}$ $\implies \alpha^{-1}(V) = h^{-1}(\Phi^{-1}(V)) \in \mathcal{F}$.

Then $h(x) = \begin{cases} 1, & x \in E \\ f(x) \neq 0, & x \notin E \end{cases} \implies \alpha(x) = \begin{cases} 1, & x \in E \\ \dfrac{f(x)}{|f(x)|} = 1, & x \notin E \end{cases} \implies f(x) = \alpha(x) \cdot |f(x)|$ $\forall x \in X$. $(x \in E \Rightarrow 0 = 0 \checkmark)$

III Borel σ-Algebra $\mathcal{B}$

$f \circ$ comp $\begin{cases} \text{cont (meas)} = \text{meas} \\ \text{borel (meas)} = \text{meas} \end{cases}$

Borel σ-Algebra = (X, τ) a top space:

The Borel σ-Algebra in X is: $\mathcal{B} = \mathcal{F}_\tau$ = minimal σ-Alg containing τ = σ-Algebra generated by τ

Borel function: A measurable funct° on $(X, \mathcal{B})$

Prop =
- $f : (X, \mathcal{B}) \to (Y, \tau)$: f continuous $\implies f$ Borel
- $F \subset X$ closed $\implies F \in \mathcal{B}$

Proof =
- Y top space : $f : X \to Y$ cont $\implies \forall$ open $V \subset Y$, then $f^{-1}(V) \in \tau \subset \mathcal{B} \implies f^{-1}(V) \in \mathcal{B}$.
- F closed $\implies F^c$ open $\implies F^c \in \mathcal{B} \implies F = (F^c)^c \in \mathcal{B}$ as $\mathcal{B}$ is a σ-Algebra

Ex:
- G_δ sets are Borel sets (countable intersect° of open sets)
- F_σ sets are Borel sets (countable union of closed sets)

Prop: $(X, \mathcal{F})$ a MSp, Y a set, $f : X \to Y$ a f°

$\mathcal{R} = \{E \subset Y : f^{-1}(E) \in \mathcal{F}\}$ is a σ-Algebra in Y

Proof =
- $f^{-1}(Y) = X \in \mathcal{F} \implies Y \in \mathcal{R}$
- if $E \in \mathcal{R}$: $f^{-1}(E) \in \mathcal{F}$ as $\mathcal{F}$ is a σ-Alg $\Rightarrow f^{-1}(E^c) = [f^{-1}(E)]^c \in \mathcal{F} \implies E^c \in \mathcal{R}$
- if $(E_n)_{n=1}^\infty \in \mathcal{R}$: $f^{-1}(E_n) \in \mathcal{F} \implies \bigcup_{n=1}^\infty f^{-1}(E_n) \in \mathcal{F}$ $\Rightarrow f^{-1}(\bigcup_{n=1}^\infty E_n) = \bigcup_{n=1}^\infty f^{-1}(E_n) \in \mathcal{F}$ $\implies \bigcup_{n=1}^\infty E_n \in \mathcal{R}$.

Prop = $(X, \mathcal{F})$ a MSp, (Y, τ) a top Space:

$f : X \to Y$ is measurable $\implies \forall A \subset Y$, A Borel (generated by τ) : $f^{-1}(A) \in \mathcal{F}$

Proof = Let $R = \{E \subset Y : f^{-1}(E) \in \mathcal{F}\}$: Any open set in Y is in $\mathcal{R}$ ((??) f measurable) $\implies \tau \subset \mathcal{R}$.
But $\mathcal{R}$ is a σ-Algebra, so $\mathcal{B} \subset \mathcal{R}$ ($\mathcal{B}$ = minimal σ-Alg containing τ)

Prop = $(X, \mathcal{F})$ a MSp, (Y, τ_Y) & (Z, τ_Z) top Spaces: $\begin{cases} f : X \to Y \text{ meas} \\ g : Y \to Z \text{ Borel} \end{cases} \implies h = g \circ f : X \to Z$ is measurable

Proof = open $V \subset Z$:
$h^{-1}(V) = f^{-1}(\underbrace{\overset{\text{Borel}}{g^{-1}(V)}}_{\text{Borel}}) \underset{\text{prev prop}}{\in} \mathcal{F} \implies h$ is meas.

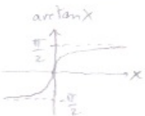
IV The Extended Real line :

Extended real line : $\mathbf{X} = [-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$, with ordering : $-\infty < a < \infty \quad \forall a \in \mathbb{R}$

Def : $0 \cdot \infty := 0$ in measure theory (but $\infty - \infty$ still undetermined!!)

Topology :
- topology τ on $X = [-\infty, \infty]$: generated by intervals : $\{[-\infty, a), (a, \infty], (a,b) : a,b \in \mathbb{R}\}$

  $[-\infty, a) = \{t \in [-\infty,\infty] : t < \alpha\}$
  $(\alpha, \infty] = \{t \in [-\infty,\infty] : t > \alpha\}$



- (X, τ) is metrizable : $d(x,y) = |\arctan x - \arctan y|$ ← $\arctan(\pm\infty) = \pm\frac{\pi}{2}$
- (X, d) is compact (but not $(\mathbb{R}, d)$!) + homeomorphic to $([-1,1], |\cdot|)$
- the induced topology on $\mathbb{R} \subset [-\infty, \infty]$ is the standard topology $\tau = \{(a,b) : a,b \in \mathbb{R}, a \leq b\}$

Lim Sup / Lim inf : $(a_n)_{n=1}^{\infty} \in [-\infty, \infty]$:

$\mathcal{L} :=$ set of all accumulat. pts of $(a_n)_{n=1}^{\infty}$, i.e. $a \in \mathcal{L} \Leftrightarrow \exists (a_{n_k})_{k=1}^{\infty}$ ST $\lim_{k \to \infty} a_{n_k} = a$

- $\mathcal{L}$ is closed , So $\begin{cases} \sup \mathcal{L} \\ \inf \mathcal{L} \end{cases}$ always exist (but can be $\pm\infty$) $\Rightarrow \begin{cases} \limsup_{n\to\infty} a_n := \sup \mathcal{L} \\ \liminf_{n\to\infty} a_n := \inf \mathcal{L} \end{cases}$ always exist (but can be $\pm\infty$)

- Prop : $\boxed{\begin{cases} \limsup_{n\to\infty} a_n = \inf_{k\geq 1} \sup_{i \geq k} a_i \\ \liminf_{n\to\infty} a_n = \sup_{k \geq 1} \inf_{i \geq k} a_i \end{cases}}$

Prop : $(X, \mathcal{F})$ a MSp , $f : X \to [-\infty, \infty]$ a f° : f is measurable $\Leftrightarrow f^{-1}((\alpha, \infty]) \in \mathcal{F} \ \forall \alpha \in \mathbb{R}$.
$\Leftrightarrow f^{-1}([-\infty, \alpha)) \in \mathcal{F} \ \forall \alpha \in \mathbb{R}$

Proof : $\boxed{\Rightarrow}$ $(\alpha, \infty]$ is open in $[-\infty, \infty]$, so f meas $\Rightarrow f^{-1}((\alpha, \infty]) \in \mathcal{F}$

$\boxed{\Leftarrow}$ $\mathcal{R} := \{E \subset [-\infty, \infty] : f^{-1}(E) \in \mathcal{F}\}$ is a σ-Algebra

Goal: $V \subset [-\infty,\infty]$ open $\Downarrow$ $f^{-1}(V) \in \mathcal{F}$

- $f^{-1}((\alpha, \infty]) \in \mathcal{F} \ \forall \alpha \in \mathbb{R} \Rightarrow (\alpha, \infty] \in \mathcal{R} \ \forall \alpha \in \mathbb{R}$
- Given $\alpha \in \mathbb{R}$: let $(\alpha_n)_{n=1}^{\infty}$ ST $\alpha_n \to \alpha$ as $n \to \infty$. Then : $[-\infty, \alpha) = \bigcup_{n=1}^{\infty} [-\infty, \alpha_n) = \underbrace{\bigcup_{n=1}^{\infty} ((\alpha_n, \infty])^c}_{\in \mathcal{R} \ (\because) \ \mathcal{R} \text{ is a } \sigma\text{-Algebra}} \in \mathcal{R}$
- $\forall \alpha \in \mathbb{R}$: $[-\infty, \alpha) \in \mathcal{R}$
- $\forall a,b \in \mathbb{R}$: $(a,b) = [-\infty, b) \cap (a, \infty] \in \mathcal{R}$ $\Big\} \Rightarrow \mathcal{R}$ contains $\mathcal{A} = \{[-\infty, a), (a, \infty], (a,b) : a,b \in \mathbb{R}\}$
- Fix an open set $V \subset [-\infty, \infty]$: $V = \bigcup_{n=1}^{\infty} A_n$ for $(A_n)_{n=1}^{\infty} \in \mathcal{A}$ (∵) any open set $= \bigcup_{n=1}^{\infty}$ of open intervals in $(-\infty, \infty)$ + deal with $\pm\infty$ cases.

$\Rightarrow \mathcal{R} \supset \tau \Rightarrow$ for any open set $V \subset [-\infty, \infty]$: $V \in \mathcal{R} \Rightarrow f^{-1}(V) \in \mathcal{F} \Rightarrow f$ is meas

Prop : $(X, \mathcal{F})$ a MSp , $f_n : \mathbf{X} \longrightarrow [-\infty, \infty]$, $n = 1, 2, \dots$ a seq of MF°

then : $\begin{cases} g = \sup_{n\geq 1} f_n \\ \tilde{g} = \inf_{n \geq 1} f_n \end{cases}$ and $\begin{cases} h = \limsup_{n\to\infty} f_n \\ \tilde{h} = \liminf_{n\to\infty} f_n \end{cases}$ are **meas**urable functions.

Note : $g = \sup_{n\geq 1} f_n \Rightarrow g(x) = \sup\{f_n(x) : n \geq 1\}$

Proof :
- claim : $g^{-1}((\alpha, \infty]) \in \mathcal{F} \ \forall \alpha \in \mathbb{R}$

  (∵) $g^{-1}((\alpha, \infty]) = \{x \in X : g(x) > \alpha\} = \{x \in X : \sup_{n\geq 1} f_n(x) > \alpha\} = \{x \in X : \exists n \in \mathbb{N} \text{ ST } f_n(x) > \alpha\} = \bigcup_{n=1}^{\infty} \underbrace{f_n^{-1}((\alpha, \infty])}_{\in \mathcal{F} \ (\because) \ f_n \text{ are MF°}} \in \mathcal{F}$
- $\tilde{g} = \inf_{n\geq 1} f_n = -\sup_{n \geq 1} (-f_n)$: $(-f_n)$ is meas (∵) map $x \mapsto -x$ is continuous $\Rightarrow$ use composit°
  $\Rightarrow \sup(-f_n)$ is meas (previous) $\Rightarrow -\sup(-f_n)$ is meas $\Rightarrow \tilde{g}$ is meas !
- $h(x) = \limsup_{n\to\infty} f_n(x) = \inf_{k\geq 1} \sup_{i\geq k} f_i(x)$ is measurable. ($\sup_{i \geq k} f_i(x) \to$ meas, $\inf_{k \geq 1} \to$ meas)
- $\tilde{h} = \liminf_{n\to\infty} f_n = -\limsup_{n\to\infty}(-f_n) \Rightarrow$ meas

Corr :
- $f_n : X \to [-\infty, \infty]$ a seq of MF° ST $\lim_{n\to\infty} f_n(x) = f(x)$ exists : then f is measurable. ($\limsup = \sup \overset{?}{=} f$)
- $f, g : X \to [-\infty, \infty]$ MF° $\Rightarrow \begin{cases} h = \max\{f, g\} \\ \tilde{h} = \min\{f, g\} \end{cases}$ are also MF° (∵) take $\begin{cases} f_1 = f \\ f_n = g \end{cases} \ \forall n \geq 2$ ✓

⚠ Wrong Proof : $\sup_{n\geq 1} f_n(x) = \max\{f, g\} = \frac{f(x) + g(x)}{2} + \frac{|f(x) - g(x)|}{2}$
works in $\mathbb{R}$ but NOT in $[-\infty, \infty]$ (∵) $\infty + (-\infty) = ???$

(V) Simple functions : $\longrightarrow$ Building block of integration theory

Meas. Simple function : $(X,\mathcal{F})$ a MSp, $S: X \longrightarrow \mathbb{C}$ (or $\mathbb{R}$) a MF° : S is simple $\iff$ the range of f consists of finitely many points.

Prop : S_1, S_2 Simple MF° $\Rightarrow \alpha S_1,\ S_1+S_2$ & $S_1 \cdot S_2$ are Simple MF°

Cannonical/Standard representation : S a Simple MF°, $\alpha_1,\dots,\alpha_n$ the distinct values of S :

then $\boxed{S = \sum_{k=1}^{n} \alpha_k \chi_{A_k}}$, where $A_k := \{x\in X : s(x) = \alpha_k\}$

Prop :
- The A_K's are nonempty, disjoint and $\bigcup_{k=1}^{n} A_k = X$
- $(X,\mathcal{F})$ a MSp : S meas $\iff$ Each A_k is measurable

($\Leftarrow$) A_k meas $\Rightarrow \chi_{A_k}$ meas $\Rightarrow \sum_{k=1}^{n} \alpha_k \chi_{A_k}$ meas

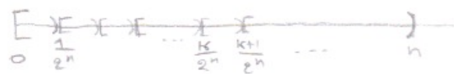
Note :
- ANY funct° of the form : $S = \sum_{k=1}^{n} \gamma_k \chi_{C_k}$ ($\gamma_k \in \mathbb{C}$ (or $\mathbb{R}$), C_k meas) is a Simple MF°.

($\Leftarrow$) the value of S lies in the set : $\left\{ \sum_{k=1}^{m} \gamma_k \chi_{C_k} : \chi_{C_k} \in \{0,1\},\ 1\le k\le n \right\}$ which is finite

- $S = \sum_{k=1}^{n} \gamma_k \chi_{C_k}$ is not necessarily the standard rep of S.

Thm : $(X,\mathcal{F})$ a MSp, $f: X \longrightarrow [0,\infty]$ a MF° :

then $\exists$ a Seq of Simple MF° $S_n : X \longrightarrow [0,\infty]$ ST $\begin{cases} (1)\ 0 \le S_1 \le S_2 \le \dots \le f \\ (2)\ \lim_{n\to\infty} S_n(x) = f(x) \quad \forall x\in X \end{cases}$

Proof : 

- **Construct S_n** :
  - divide $[0,n)$ into segments $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$, $0 \le k \le 2^n\cdot n - 1$ (of length $\frac{1}{2^n}$)
  - Given n : $A_{n,k} = \left\{x\in X : \frac{k}{2^n} \le f(x) < \frac{k+1}{2^n}\right\} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right) \leadsto A_{n,k}$ is measurable as inverse image of a Borel set

    $B_n = \{x\in X : f(x) \ge n\} = f^{-1}([n,\infty]) \leadsto B_n$ is measurable as

    $\boxed{S_n = \sum_{k=0}^{2^n n-1} \frac{k}{2^n} \chi_{A_{n,k}} + n\chi_{B_n}}$ $\Rightarrow$ obviously : $\begin{cases} S_n \text{ is a simple MF°} \\ S_n(x) \le f(x)\ \forall x\in X. \end{cases}$

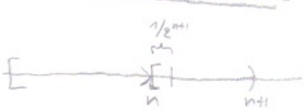
- **$\lim_{n\to\infty} S_n(x) = f(x)\ \forall x\in X$** :
  - if $f(x) = \infty$: then $x \in B_n\ \forall n$, so $S_n(x) = n\ \forall n \Rightarrow \lim_{n\to\infty} S_n(x) = \infty = f(x)\ \forall x\in X$
  - if $f(x) < \infty$: say $f(x) < N$ for some $N>0$: then $\forall n > N$, $\exists k \in \mathbb{N}$ ST $\frac{k}{2^n} \le f(x) \le \frac{k+1}{2^n}\ \forall x\in A_{n,k}$

    In this case, $S_n(x) = \frac{k}{2^n} \Rightarrow |f(x) - S_n(x)| \le \frac{1}{2^n} \Rightarrow \lim_{n\to\infty} S_n(x) = f(x)$

    Note : if f is bounded, then we can choose N ST : $f(x) < N\ \forall x \Rightarrow |f(x) - S_n(x)| \le \frac{1}{2^n}$ holds $\forall x$

    $\hookrightarrow$ the convergence is uniform

- **$S_n \le S_{n+1}\ \forall n\in\mathbb{N}$** :
  - Case 1 : $x\in B_n \Rightarrow f(x) \ge n \Rightarrow S_n = n$

    - if $x \in B_{n+1}$: $f(x) \ge n+1 \Rightarrow S_{n+1}(x) = n+1 > n = S_n(x)$
    - if $n \le f(x) < n+1$: then $\exists k\in\mathbb{N}$ ST $\frac{k}{2^{n+1}} \le f(x) < \frac{k+1}{2^{n+1}} \Rightarrow k \ge 2^n\cdot n \Rightarrow S_{n+1}(x) \ge \frac{k}{2^{n+1}} \ge n = S_n(x)$
  - Case 2 : $x \in A_{n,k}$, $k \le 2^{n+1}\cdot n - 1 \Rightarrow \frac{k}{2^n} \le f(x) < \frac{k+1}{2^n} \Rightarrow S_n(x) = \frac{k}{2^n}$

    Also : $\frac{2k}{2^{n+1}} \le f(x) \le \frac{2k+2}{2^{n+1}}$, $2k < 2^{n+1}\cdot n - 2 < 2^{n+1}(n+1) - 1$

    $\Rightarrow S_{n+1}(x) = \begin{cases} \frac{2k}{2^{n+1}} , & \frac{2k}{2^{n+1}} \le f(x) < \frac{2k+1}{2^{n+1}} \\ \frac{2k+1}{2^{n+1}} , & \frac{2k+1}{2^{n+1}} \le f(x) < \frac{2k+2}{2^{n+1}} \end{cases}$ $\Rightarrow S_{n+1}(x) = \frac{k}{2^n}$ or $\frac{k}{2^n} + \frac{1}{2^{n+1}}$

    $\Rightarrow S_{n+1}(x) \ge \frac{k}{2^n} = S_n(x)$ in both cases.

VI Measures :

Measure : $(X, \mathcal{F})$ a MSp : The function $\mathcal{M} : \mathcal{F} \longrightarrow [0, \infty]$ is a measure on $(X, \mathcal{F})$

if : $\forall (A_n)_{n=1}^{\infty} \in \mathcal{F}$ of disjoint MSets ($\to A_n \cap A_m = \emptyset \ \forall n \neq m$) : $\boxed{\mathcal{M}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathcal{M}(A_n)}$

Note : • $(X, \mathcal{F}, \mathcal{M})$ is a measure space

• If $\mathcal{M}(X) = 1$: $\mathcal{M}$ is a probability measure (denoted "p") & $(X, \mathcal{F}, \mathcal{M})$ is a probability space

• Always assume $\mathcal{M}$ to be Non trivial : $\exists A \in \mathcal{F}$ ST $\mathcal{M}(A) < \infty$.

Theorem : If $(X, \mathcal{F}, \mathcal{M})$ is a Measure Space :

A is an MSet $= A \in \mathcal{F}$

(1) $\mathcal{M}(\emptyset) = 0$

(2) If $A_1, \dots, A_n \in \mathcal{F}$ are disjoint $\Rightarrow \mathcal{M}(A_1 \cup \dots \cup A_n) = \mathcal{M}(A_1) + \dots + \mathcal{M}(A_n)$

(3) If $A, B \in \mathcal{F}$: $A \subset B \Rightarrow \mathcal{M}(A) \leq \mathcal{M}(B)$

(4) If $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots$ is an increasing seq of MSets : $A = \bigcup_{n=1}^{\infty} A_n \Longrightarrow \mathcal{M}(A) = \lim_{n\to\infty} \mathcal{M}(A_n)$

(5) If $A_1 \supset A_2 \supset \dots \supset A_n \supset \dots$ is a decreasing Seq of M sets : $A = \bigcap_{n=1}^{\infty} A_n$, $\mathcal{M}(A_1) < \infty \Rightarrow \mathcal{M}(A) = \lim_{n\to\infty} \mathcal{M}(A_n)$

(6) If $(A_n)_{n=1}^{\infty} \in \mathcal{F}$: $\mathcal{M}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mathcal{M}(A_n)$

Proof :

(1) Let $A \in \mathcal{F}$, $\mathcal{M}(A) < \infty$: let $\begin{cases} A_1 = A \\ A_n = \emptyset, n \geq 2 \end{cases} \Rightarrow \mathcal{M}(A) = \mathcal{M}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathcal{M}(A_n) = \mathcal{M}(A) + \sum_{n=2}^{\infty} \mathcal{M}(\emptyset) \geq \mathcal{M}(A) + \mathcal{M}(\emptyset)$
$\Rightarrow 0 \geq \mathcal{M}(\emptyset) \geq 0$

(2) $A_1, \dots, A_n \in \mathcal{F}$ disjoint : let $A_{n+1} = A_{n+2} = \dots = \emptyset$, then : $\mathcal{M}\left(\bigcup_{k=1}^{n} A_k\right) = \mathcal{M}\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mathcal{M}(A_k) = \sum_{k=1}^{n} \mathcal{M}(A_k)$

(A) (B)

(3) $A, B \in \mathcal{F}$, $A \subset B$: $B = B \cap (A \cup A^c) = (A \cap B) \cup (A^c \cap B) = A \cup (B \setminus A) \Rightarrow \mathcal{M}(B) = \mathcal{M}(A) + \underbrace{\mathcal{M}(B \setminus A)}_{\geq 0} \geq \mathcal{M}(A)$

(4) set $\begin{cases} B_1 = A_1 \\ B_n = A_n \setminus A_{n-1} = A_n \cap A_{n-1}^c, \forall n \geq 2 \end{cases}$

• B_n's are measurable and disjoint : (*) $B_n \subset A_n \subset A_{m-1}$ $\forall m > n$, and $B_m \cap A_{m-1} = \emptyset \Rightarrow B_n \cap B_m = \emptyset \ \forall n \neq m$

• $\forall n$: $A_n = A_1 \cup \dots \cup A_n = B_1 \cup \dots \cup B_n$ ((*) Inductº : $n=1$ ✓ ; If $A_n = B_1 \cup \dots \cup B_n$, then $B_1 \cup \dots \cup B_n \cup B_{n+1} = A_n \cup B_{n+1} = A_n \cup (A_{n+1} \setminus A_n)$
$= A_{n+1}$ (*) $A_n \subset A_{n+1}$)

• So $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$ (*) $A_k \subset B_k \overset{\forall k}{\Rightarrow} A_k \subset \bigcup_{m=1}^{\infty} B_m \ \forall k \Rightarrow \bigcup_{k=1}^{\infty} A_k \subset \bigcup_{m=1}^{\infty} B_n$; $B_k \subset A_k \ \forall k \Rightarrow B_k \subset \bigcup_{m=1}^{\infty} A_m \ \forall k$
$\Rightarrow \bigcup_{k=1}^{\infty} B_k \subset \bigcup_{n=1}^{\infty} A_n$

$\Rightarrow$ $\mathcal{M}(A) = \mathcal{M}\left(\bigcup_{n=1}^{\infty} A_n\right) = \mathcal{M}\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mathcal{M}(B_n) = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \mathcal{M}(B_k)\right) = \lim_{n\to\infty} \mathcal{M}\left(\bigcup_{k=1}^{n} B_k\right) = \lim_{n\to\infty} \mathcal{M}(A_n)$

(5) Set : $\begin{cases} C_1 = A_1 \\ C_n = A_1 \setminus A_n = A_1 \cap A_n^c, \forall n \geq 2 \end{cases}$

• $(A_n)_{n=1}^{\infty}$ is ↘ $\Rightarrow (A_n^c)_{n=1}^{\infty}$ is ↗ $\Rightarrow (C_n)_{n=1}^{\infty}$ is a ↗ seq of MSets $\overset{(4)}{\Rightarrow} \mathcal{M}\left(\bigcup_{n=1}^{\infty} C_n\right) = \lim_{n\to\infty} \mathcal{M}(C_n)$

• $\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} (A_1 \cap A_n^c) = A_1 \cap \left(\bigcup_{n=1}^{\infty} A_n^c\right) = A_1 \cap \left(\bigcap_{n=1}^{\infty} A_n\right)^c = A_1 \cap A^c = A_1 \setminus A \Rightarrow \mathcal{M}(A_1 \setminus A) = \lim_{n\to\infty} \mathcal{M}(C_n)$

• $A \subset A_1 \Rightarrow A_1 = A_1 \cup (A \setminus A_1) \Rightarrow \mathcal{M}(A_1) = \mathcal{M}(A) + \mathcal{M}(A_1 \setminus A) \overset{\mathcal{M}(A_1) < \infty}{\Longrightarrow} \mathcal{M}(A_1) - \mathcal{M}(A) = \mathcal{M}(A_1 \setminus A) = \lim_{n\to\infty} \mathcal{M}(C_n)$

• $A_n \subset A_1 \Rightarrow C_n = A_1 \cap A_n^c = A_1 \setminus A_n \Rightarrow \mathcal{M}(C_n) = \mathcal{M}(A_1) - \mathcal{M}(A_n) \Longrightarrow \mathcal{M}(A_1) - \mathcal{M}(A) = \lim_{n\to\infty} (\mathcal{M}(A_1) - \mathcal{M}(A_n))$

$\Rightarrow$ $\mathcal{M}(A_1) < \infty \Rightarrow \mathcal{M}(A_1) - \mathcal{M}(A) = \mathcal{M}(A_1) - \lim_{n\to\infty} \mathcal{M}(A_n) \Rightarrow \lim_{n\to\infty} \mathcal{M}(A_n) = \mathcal{M}(A)$

(6) (HW2) Hint : Set $\begin{cases} B_1 = A_1 \\ B_n = A_n \setminus (B_1 \cup \dots \cup B_{n-1}) \end{cases}$: B_n's measurable + disjoint $\Rightarrow A_1 \cup \dots \cup A_n = B_1 \cup \dots \cup B_n \overset{\text{Indº}}{\Longrightarrow} \mathcal{M}(B_n) \leq \mathcal{M}(A_n)$

Examples : X a set, $\mathcal{F} = \mathcal{P}(X)$:

• Dirac Unit mass measure at x_0 : $x_0 \in X$, $\mathcal{M}(E) := \begin{cases} 1, & x_0 \in E \\ 0, & x_0 \notin E \end{cases}$ is a measure

↳ like the delta functº δ_{x_0} in physics : $\int_X f \, d\delta_{x_0} = f(x_0)$

(*) (E_n) disjoint $\Rightarrow \mathcal{M}\left(\bigcup_{n=1}^{\infty} E_n\right) \overset{?}{=} \sum_{n=1}^{\infty} \mathcal{M}(E_n)$

YES : if $x_0 \in \bigcup E_n \Rightarrow \mathcal{M}(\bigcup E_n) = 1$, $x_0 \in E_{n_0} \Rightarrow \sum \mathcal{M} = 1$

if $x_0 \notin \bigcup E_n \Rightarrow \mathcal{M}(\bigcup E_n) = 0$, $x_0 \notin E_n \forall n \Rightarrow \sum \mathcal{M} = 0$

• Counting measure : $\mathcal{M}(E) = \#$ of elements in $E \subset X$ is a measure

(*) $\mathcal{M}$ counts the nb of elements $\Rightarrow \mathcal{M}(\bigcup E_n) = \sum \mathcal{M}(E_n)$

⚠ Ex : $X = \mathbb{N}$, $A_n = \{n+1, n+2, \dots\}$, $\mathcal{M}$ = counting meas

So : $\begin{cases} A_1 \supset A_2 \supset \dots \supset A_n \supset \dots \Rightarrow \bigcap_{n=1}^{\infty} A_n = \emptyset \Rightarrow \mathcal{M}\left(\bigcap_{n=1}^{\infty} A_n\right) = 0 \\ \mathcal{M}(A_n) = \infty \ \forall n \Rightarrow \lim_{n\to\infty} \mathcal{M}(A_n) = \infty \end{cases} \Rightarrow \mathcal{M}\left(\bigcap_{n=1}^{\infty} A_n\right) = 0 \neq \infty = \lim_{n\to\infty} \mathcal{M}(A_n) \to$ (5) Breaks

(*) $\mathcal{M}(A_1) = \infty$

VIII Link with Probability theory:

Probability Space: $(X, \mathcal{F}, \mu)$ a Measure Space, $\mu(X) = 1$ (μ: probability measure)

Note: Mathematical Structure Measure = Mathematical structure probability theory — Intuition

Events: • Measurable sets are called Events in prob theory: $E \in \mathcal{F}$ is an event
• Elementary events = Points $x \in X$ → prob: $\omega \in \Omega$

Probability: • Probability of event E ($E \in \mathcal{F}$) $= \mu(E)$
• $\mu(E_1 \cup E_2)$ = Prob that either E_1 or E_2 happens
• $\mu(E_1 \cap E_2)$ = prob that both E_1 and E_2 happen

Independance: $(X, \mathcal{F}, \mu)$, $\mu(X) = 1$ a prob space:

$E_1, E_2 \in \mathcal{F}$ are independant $\iff$ $\boxed{\mu(E_1 \cap E_2) = \mu(E_1)\mu(E_2)}$

Liminf: $(E_n)_{n=1}^{\infty} \in \mathcal{F}$ a seq of events (i.e. M sets): $\boxed{\liminf_{n\to\infty} E_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k}$

(ఠ) $x \in \liminf_{n\to\infty} E_n \Rightarrow \exists$ a seq $(n_k)_{k=1}^{\infty}$ (may depend on X) $: n_1 < n_2 < \dots < n_k < \dots$ ST $x \in E_{n_k}$

$\Rightarrow \liminf_{n\to\infty} E_n$ = event that ∞ many events E_n take place

Borel-Cantelli lemma: $\sum_{k=1}^{\infty} \mu(E_k) < \infty \Longrightarrow \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right) = 0$

Proof: "If the sum of probs of Events E_n is finite, then, with prob 1: only finitely many events can occur simultaneously"

$\forall N > 0$: $A := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \subset \bigcup_{k=N}^{\infty} E_k \Rightarrow \mu(A) \leq \mu\left(\bigcup_{k=N}^{\infty} E_k\right) \underset{HW2}{\leq} \sum_{k=N}^{\infty} \mu(E_k)$

But $\sum_{k=N}^{\infty} \mu(E_k) < \infty \Rightarrow \lim_{N\to\infty} \sum_{k=N}^{\infty} \mu(E_k) = 0 \Rightarrow \mu(A) = 0$

Note: $\sum_{k=1}^{\infty} \mu(E_k) = \infty \not\Rightarrow \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right) = 1$ in general

Ex: $E_n = [0, \frac{1}{n}]$, $X = [0,1]$, μ = length (E_n)

Borel 0-1 law: Need $(E_k)_{k=1}^{\infty}$ indep!

Integration of positive functions :

Remarks on $[0,\infty]$:
- $[0,\infty]$ with $d(x,y)=|\arctan x - \arctan y|$ is topologically equivalent to $([0,\infty), |\cdot|)$
- Define : $a+\infty=\infty$ $(0\le a\le\infty)$, $a\cdot\infty = \begin{cases}\infty, & 0<a\le\infty\\ 0, & a=0\end{cases}$
- $a+b$ & $a\cdot b$ are still commutative, associative and distributive
- Addition is continuous : $+: [0,\infty]\times[0,\infty]\to[0,\infty]$, $(x,y)\mapsto x+y$ is continuous

  Corr : $f,g : X\to[0,\infty]$ MF° $\Rightarrow$ $f+g : X\to[0,\infty]$ MF°
- ⚠ Multiplicat° is NOT continuous : $\cdot : [0,\infty]\times[0,\infty]\to[0,\infty]$, $(x,y)\mapsto x\cdot y$ is NOT cont

  (∞) Ex : $\begin{cases}x_n=\infty\\ y_n=\frac{1}{n}\end{cases}$ $\forall n$ $\Rightarrow \begin{cases}x_n y_n=\infty\\ \lim y_n = 0\end{cases}$ So : $\lim(x_n y_n)=\infty\ne 0=\lim x_n\cdot\lim y_n$

1) Integration of Simple functions :

Prop : Let $\begin{cases}0\le x_1\le x_2\le\cdots\\ 0\le y_1\le y_2\le\cdots\end{cases}$ two increasing seq in $[0,\infty]$: then $\lim\limits_{n\to\infty}(x_n y_n) = \left(\lim\limits_{n\to\infty}x_n\right)\left(\lim\limits_{n\to\infty}y_n\right)$

Corr : $f,g : X\to[0,\infty]$ MF° $\Rightarrow$ $f\cdot g : X\times X\to[0,\infty]$ is a MF°

Proof : Let $\begin{cases}s_n\\ t_n\end{cases}$ be an ↗ seq of nonnegative simple MF°, ST : $\begin{cases}s_n\to f\\ t_n\to g\end{cases}$

then $\lim\limits_{n\to\infty}(s_n t_n)=fg$ in $[0,\infty]$, So : $s_n t_n$ MF° $\Rightarrow fg$ MF°

Standard representation : $s : X\to[0,\infty]$ a simple function : $s=\sum\limits_{k=1}^{n}\alpha_k\chi_{A_k}$ where $\begin{cases}\alpha_k = \text{distinct values of } s\\ A_k = \text{disjoint, measurable, } X=\bigcup\limits_{k=1}^{n}A_k\end{cases}$

↳ unique !

Regular representation : $s : X\to[0,\infty]$ a simple function : $s=\sum\limits_{k=1}^{m}\gamma_k\chi_{B_k}$ where $\begin{cases}\gamma_k = \text{not necessarily distinct}\\ B_k = \text{disjoint, measurable, } X=\bigcup\limits_{k=1}^{m}B_k\end{cases}$

↳ NOT unique

Lebesgue integral : Let $(X,\mathcal{F},\mathcal{M})$, $E\in\mathcal{F}$ a Mset, $s : X\to[0,\infty]$ a simple MF°, $s=\sum\limits_{k=1}^{m}\gamma_k\chi_{B_k}$

the Lebesgue integral of s over E is : $\boxed{I_E(s)=\int_E s\,d\mu=\sum\limits_{k=1}^{m}\gamma_k\,\mu(B_k\cap E)}$ (regular rep, $\gamma_k\ge0$)

Prop : $I_E(s)$ does not depend on the choice of regular rep of s

Proof : Let $s=\sum\limits_{k=1}^{m}\gamma_k\chi_{B_k}=\sum\limits_{j=1}^{m'}\gamma_j'\chi_{B_j'}$ be 2 reg. rep. of s

if $B_k\cap B_j'\ne\emptyset$: $\forall x\in B_k\cap B_j'$, $s(x)=\gamma_j'=\gamma_k$

So $\sum\limits_{k=1}^{m}\gamma_k\mathcal{M}(B_k\cap E)=\sum\limits_{k=1}^{m}\gamma_k\mathcal{M}(B_k\cap E\cap X)=\sum\limits_{k=1}^{m}\gamma_k\mathcal{M}\left(B_k\cap E\cap\left(\bigcup\limits_{j=1}^{m'}B_j'\right)\right)$

$=\sum\limits_{k=1}^{m}\gamma_k\mathcal{M}\left(\bigcup\limits_{j=1}^{m'}(B_j'\cap B_k\cap E)\right)=\sum\limits_{k=1}^{m}\gamma_k\sum\limits_{j=1}^{m'}\mathcal{M}(B_j'\cap B_k\cap E)=\sum\limits_{j=1}^{m'}\gamma_j'\sum\limits_{k=1}^{m}\mathcal{M}(B_k\cap B_j'\cap E)$ ($\gamma_j'=\gamma_k$ on $B_k\cap B_j'$)

$=\sum\limits_{j=1}^{m'}\gamma_j'\mathcal{M}\left(\left(\bigcup\limits_{k=1}^{m}B_k\right)\cap E\cap B_j'\right)=\sum\limits_{j=1}^{m'}\gamma_j'\mathcal{M}(E\cap B_j')$

Prop :
- $\int_E cs\,d\mu = c\int_E s\,d\mu$ $\forall c\ge0$ cst
- $E_1\subset E_2 \Rightarrow \int_{E_1}s\,d\mu\le\int_{E_2}s\,d\mu$ (bc) $\mathcal{M}(B_k\cap E_1)\le\mathcal{M}(B_k\cap E_2)$

Note : $I_E(s)=\sum\limits_{k=1}^{m}\gamma_k\mathcal{M}(B_k\cap E)$ could be $+\infty$ (e.g. : $\mathcal{M}(B_k\cap E)=\infty$, $\gamma_k\ne0$)

RK : Interchanging the sum :

$a_{ij}\in[0,\infty] \Rightarrow \sum\limits_{j=1}^{\infty}\sum\limits_{i=1}^{\infty}a_{ij}=\sum\limits_{i=1}^{\infty}\sum\limits_{j=1}^{\infty}a_{ij}$ (Need $a_{ij}\ge0$!!) → Proof : HW 2, #2

Finite sums : Fubini thm for counting measure

↳ extend to ∞ sums !

$\begin{bmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{n1} & \cdots & a_{nn}\end{bmatrix} \Rightarrow \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}a_{ij}=\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{n}a_{ij}$

Prop = $(X,\mathcal{F},\mu)$: $E\in\mathcal{F}$, $\boxed{\Phi(E) := \int_E s\,d\mu}$ for some simple MF° $s: X\to[0,\infty)$

then Φ is a measure on $(X,\mathcal{F})$

Proof = Let reg rep of $s=\sum_{k=1}^{m}\gamma_k \chi_{B_k}'$, $(E_n)_{n=1}^{\infty}$ a seq of disjoint Msets.

interchange sum!!

$$\Phi\left(\bigcup_{n=1}^{\infty}E_n\right)=\int_{\bigcup_{n=1}^{\infty}E_n} s\,d\mu=\sum_{k=1}^{m}\gamma_k\,\mu\left(B_k\cap\left(\bigcup_{n=1}^{\infty}E_n\right)\right)=\sum_{k=1}^{m}\gamma_k\sum_{n=1}^{\infty}\mu(B_k\cap E_n)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\gamma_k\,\mu(B_k\cap E_n)=\sum_{n=1}^{\infty}\Phi(E_n)$$

so $\Phi\left(\bigcup_{n=1}^{\infty}E_n\right)=\sum_{n=1}^{\infty}\Phi(E_n)$ ✓

Corr = $E_1, E_2\in\mathcal{F}$ disjoint $\Rightarrow$ $\boxed{\int_{E_1\cup E_2} s\,d\mu=\int_{E_1}s\,d\mu+\int_{E_2}s\,d\mu}$

(*) Same proof type.

Prop = $(X,\mathcal{F},\mu)$ a MSp: $s,t: X\to[0,\infty]$ two simple MF°, $E\in\mathcal{F}$

then: $\boxed{\int_E (s+t)\,d\mu=\int_E s\,d\mu+\int_E t\,d\mu}$

Proof = Let $\begin{cases} s=\sum_{k=1}^{m}\gamma_k\chi_{B_k}\\ t=\sum_{j=1}^{n}r_j\chi_{D_j}\end{cases}$ reg. rep. of s & t (i.e. the B_k's and D_j's are each disjoint)

- $(s+t)$ traces the value (γ_k+r_j) on $(B_k\cap D_j)$. ($\because$) $B_k\cap D_j$, $1\le k\le m$, $1\le j\le n$, are disjoint M sets, ~~their~~ union is X.

$\Rightarrow (s+t)=\sum_{k=1}^{m}\sum_{j=1}^{n}(\gamma_k+r_j)\chi_{B_k\cap D_j}$ (Note = we don't need $B_k\cap D_j\neq\emptyset$ since $\chi_\emptyset=0$)

$$\Rightarrow \int_E (s+t)\,d\mu=\sum_{k=1}^{m}\sum_{j=1}^{n}(\gamma_k+r_j)\,\mu(E\cap B_k\cap D_j)=\sum_{k=1}^{m}\sum_{j=1}^{n}\gamma_k\,\mu(E\cap B_k\cap D_j)+\sum_{k=1}^{m}\sum_{j=1}^{n}r_j\,\mu(E\cap B_k\cap D_j)$$

$$=\sum_{k=1}^{m}\gamma_k\,\mu\Big(E\cap B_k\cap\underbrace{\Big(\bigcup_{j=1}^{n}D_j\Big)}_{=X}\Big)+\sum_{j=1}^{n}r_j\,\mu\Big(E\cap\underbrace{\Big(\bigcup_{k=1}^{m}B_k\Big)}_{=X}\cap D_j\Big)$$

$$=\int_E s\,d\mu+\int_E t\,d\mu$$

Prop = Let $s,t: X\to[0,\infty]$ be two simple MF° :

$$0\le s\le t \Rightarrow \int_E s\,d\mu\le\int_E t\,d\mu \quad \forall E\in\mathcal{F}$$

Proof = $t=(t-s)+s$: $(t-s)$ is a simple MF°, $t-s\ge 0$ $\underset{\text{Prev prop}}{\Longrightarrow}$ $\int_E t\,d\mu=\underbrace{\int_E \underbrace{(t-s)}_{\ge 0}\,d\mu}_{\ge 0}+\int_E s\,d\mu\ge\int_E s\,d\mu$.

Integral of Positive Measurable functions: $f: X \to [0,\infty]$

I) Definition & basic properties:

Lebesgue integral = $(X, \mathcal{F}, \mu)$ a MSp, $E \in \mathcal{F}$: $\boxed{\int_E f\, d\mu = \sup_{0 \le s \le f} \int_E s\, d\mu}$

where the Supremum is taken over all simple functions $0 \le s \le f$.

Note =
- Agrees with previous def: f simple, $0 \le s \le f \Rightarrow \int_E s\, d\mu \le \int_E f\, d\mu \Rightarrow \sup \int_E s\, d\mu = \int_E f\, d\mu$.
- $(X, \mathcal{F}, \mu)$ a probability space: $\begin{cases} f: X \to [0,\infty] \text{ is a random variable} \\ \int_X f\, d\mu = E[f] \text{ is the expectation value of } f. \end{cases}$

Prop = $(X, \mathcal{F}, \mu)$ a MSp, $E \in \mathcal{F}$:

- $0 \le f \le g \Rightarrow \int_E f\, d\mu \le \int_E g\, d\mu$

  ($\because$) $\begin{cases} A = \{s : 0 \le s \le f, s \text{ simple}\} \\ B = \{s : 0 \le s \le g, s \text{ simple}\} \end{cases}$: $f \le g \Rightarrow A \subset B$

  So $\int_E f\, d\mu = \sup_{s \in A} \int_E s\, d\mu \le \sup_{s \in B} \int_E s\, d\mu = \int_E g\, d\mu$

- $E_1, E_2 \in \mathcal{F}$: $f \ge 0$, $E_1 \subset E_2 \Rightarrow \int_{E_1} f\, d\mu \le \int_{E_2} f\, d\mu$

  ($\because$) $0 \le s \le f$ simple $\Rightarrow \int_{E_1} s\, d\mu \le \int_{E_2} s\, d\mu$ → take Sup on both Sides

- $\forall c \ge 0$: $\int_E c f\, d\mu = c \int_E f\, d\mu$

  ($\because$) $c \int_E f\, d\mu = c \cdot \sup \int_E s\, d\mu = \sup \left(c \int_E s\, d\mu \right) = \sup \int_E (cs)\, d\mu$

  $c > 0$, So: a simple MF° s verifies $0 \le s \le f \Leftrightarrow 0 \le cs \le cf$

  So $c \int_E f\, d\mu = \int_E c f\, d\mu$

- $f(x) = 0 \ \forall x \in E \Rightarrow \int_E f\, d\mu \overset{\text{even if } \mu(E) = \infty}{=} 0$

  ($\because$) s simple MF°: $0 \le s \le f \Rightarrow s = 0 \cdot \chi_X \Rightarrow \int_E s\, d\mu = 0 \cdot \mu(X \cap E) = 0 \cdot \mu(E) = 0$

- $\mu(E) = 0 \Rightarrow \int_E f\, d\mu \overset{\text{even if } f(x) = \infty\ \forall x \in E}{=} 0$

  ($\because$) s Simple MF°: $0 \le s \le f \Rightarrow \int_E s\, d\mu = \sum_{k=1}^{N} \alpha_k \mu(E \cap B_k) \le \sum_{k=1}^{N} \alpha_k \mu(E) = \sum 0 = 0$ $\Rightarrow \int_E f\, d\mu = \sup \int_E s\, d\mu = \sup(0) = 0$

  So $\int_E s\, d\mu = 0 \ \forall\, 0 \le s \le f \Rightarrow \sup(\cdot) = 0$

II) Lebesgue Monotone Convergence Theorem (LMCT):

LMCT: $f_n: X \to [0,\infty]$ a Seq of MF°, $\begin{cases} 0 \le f_1(x) \le f_2(x) \le \dots \le f_n(x) \le \dots \quad \forall x \in X. \\ f(x) := \lim_{n\to\infty} f_n(x) \end{cases}$

($\because$) Always exists: increasing Seq of real nb ↳ but can be ∞

Then: $\boxed{\lim_{n\to\infty} \int_E f_n\, d\mu = \int_E \left(\lim_{n\to\infty} f_n \right) d\mu = \int_E f\, d\mu}$

Proof = Fix $E \in \mathcal{F}$:

- f is a MF° ($\because$) $f = \lim_{n\to\infty} f_n = \liminf_{n\to\infty} f_n$ which is measurable as f_n are MF°
- $\lim_{n\to\infty} \int_E f_n\, d\mu$ exists ($\because$) $f_n \le f_{n+1} \Rightarrow \int_E f_n\, d\mu \le \int_E f_{n+1}\, d\mu$ is an increasing seq of real nb $\Rightarrow$ Converges in $[0,\infty]$ (WRT arctan metric)
- $\lim_{n\to\infty} \int_E f_n\, d\mu \le \int_E f\, d\mu$ ($\because$) $f_n \le f \Rightarrow \int_E f_n\, d\mu \le \int_E f\, d\mu$ So take $\lim_{n\to\infty}$ on both sides.

GOAL: $\int_E f\, d\mu \le \lim_{n\to\infty} \int_E f_n\, d\mu$

($\because$) * s a Simple MF°, $0 \le s \le f$, $0 < c < 1$, $E_n = \{x \in E : f_n(x) \ge c \cdot s(x)\}$

→ $E_n \subset E_{n+1}$ ($\because$) $f_n \le f_{n+1}$

→ E_n are MSets ($\because$) $E_n = E \cap h_n^{-1}(\underbrace{[0,\infty]}_{\text{Borel}})$ with $h_n = f_n - c \cdot s(x)$ a MF°

* claim: $\bigcup_{n=1}^{\infty} E_n = E$

→ $\bigcup_{n=1}^{\infty} E_n \subset E$ since $E_n = E \cap h_n^{-1}([0,\infty]) \subset E \quad \forall n \in \mathbb{N}$.

→ $\bigcup_{n=1}^{\infty} E_n \supset E$: (∗) Let $\underline{\underline{x \in E}}$:
- if $f(x)=0$, then $f_n(x)=0 \ \forall n \Rightarrow S(x)=0 \Rightarrow x \in E_1 \Rightarrow \underline{\underline{x \in \bigcup_{n=1}^{\infty} E_n}}$
- if $f(x)>0$, then $f(x) > c f(x) \geq c\cdot S(x) \geq 0$ as $0<c<1$.

So $\lim_{n\to\infty} f_n(x) = f(x) > c\cdot S(x) \Rightarrow \exists n \in \mathbb{N}$ ST $f_n(x) \geq c\cdot S(x) \Rightarrow x \in E_n$ for some $n \in \mathbb{N}$.
$\Rightarrow \underline{\underline{x \in \bigcup_{n=1}^{\infty} E_n}}$

* $\int_E f_n \, d\mu \underset{E_n \subset E}{\geq} \int_{E_n} f_n \, d\mu \underset{E_n = \{\cdots\}}{\geq} \int_{E_n} c\cdot s \, d\mu = c \int_{E_n} s \, d\mu$

* $\phi(A) = \int_A s \, d\mu$, $A \in \mathcal{F}$ is a measure on $(X,\mathcal{F})$; $E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots$ and $\bigcup_{n=1}^{\infty} E_n = E \Rightarrow \boxed{\lim_{n\to\infty} \phi(E_n) = \phi(E)}$

$\int_E f_n \, d\mu \geq c\cdot\phi(E_n) \Rightarrow \lim_{n\to\infty} \int_E f_n \, d\mu \geq c\cdot \lim_{n\to\infty} \phi(E_n) = c\cdot\phi(E) = c\int_E s \, d\mu \quad \forall 0<c<1$

So $\lim_{n\to\infty} \int_E f_n \, d\mu \geq c\int_E s \, d\mu \xRightarrow{c\to 1} \lim_{n\to\infty} \int_E f_n \, d\mu \geq \int_E s \, d\mu \quad \forall$ simple MF° $0 \leq s \leq f$

So $\lim_{n\to\infty} \int_E f_n \, d\mu \geq \sup_{0\leq s\leq f} \int_E s \, d\mu = \int_E f \, d\mu$

RK: The proof depends on the continuity property : for any measure μ, and ↑ seq of M sets $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$

$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$

So $\boxed{\text{LMCT} \Leftrightarrow \text{continuity property } \forall \mu \text{ on } (X,\mathcal{F})}$

(∗) ⇐ see LMCT proof

⇒ set $f_n = \chi_{A_n}$: $A_n \subset A_{n+1} \Rightarrow f_n \leq f_{n+1}$, so $\lim_{n\to\infty} f_n(x)$ exists (↑ seq of real nb)

So $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \chi_{A_n} = \chi_A$ where $A = \bigcup_{n=1}^{\infty} A_n$

(∗∗) if $x \in A$: $\chi_A = 1$ and $x \in A_{n_0} \Rightarrow \lim_{n\to\infty} \chi_{A_n} = 1$ ($\subset A_n \ \forall n \geq n_0 \Rightarrow \chi_{A_n} = 1$)
if $x \notin A$: $\chi_A = 0$ and $x \notin A_n \ \forall n \Rightarrow \chi_{A_n} = 0$ ✓

So $\lim_{n\to\infty} \int_X f_n \, d\mu = \lim_{n\to\infty} \int_X \chi_{A_n} \, d\mu \underset{\text{LMCT}}{=} \int_X \lim_{n\to\infty} \chi_{A_n} \, d\mu = \int_X \chi_A \, d\mu \Rightarrow \boxed{\lim_{n\to\infty} \mu(A_n) = \mu(A)}$

Conseq: $f : X \to [0,\infty]$ a MF°, $E \in \mathcal{F} \Rightarrow \boxed{\int_X (\chi_E f)\, d\mu = \int_E f \, d\mu}$

Proof: Let $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq \cdots \leq f$ a seq of simple MF°
ST $\lim_{n\to\infty} s_n(x) = f(x) \quad \forall x \in X$ (∗) We constructed such a seq last class

LMCT ⇒ $\int_E f \, d\mu = \lim_{n\to\infty} \int_E s_n \, d\mu$

claim: $\int_E s_n \, d\mu = \int_X (\chi_E s_n)\, d\mu$

(∗) Let $s_n = \sum_{k=1}^{m} \gamma_k \chi_{B_k}$ a reg rep of $s_n \Rightarrow \int_E s_n \, d\mu = \sum_{k=1}^{m} \gamma_k \mu(B_k \cap E)$

Also: $\chi_E \cdot s_n = \sum_{k=1}^{m} \gamma_k \chi_E \chi_{B_k} = \sum_{k=1}^{m} \gamma_k \chi_{E\cap B_k} + 0\cdot\chi_{E^c}$ is a reg. rep of $\chi_E \cdot s_n$.

$\Rightarrow \int_X \chi_E s_n \, d\mu = \sum_{k=1}^{m} \gamma_k \mu(\underbrace{X \cap E}_{=E} \cap B_k) = \int_E s_n \, d\mu \Rightarrow \boxed{\lim_{n\to\infty} \int_X \chi_E \cdot s_n \, d\mu = \lim_{n\to\infty} \int_E s_n \, d\mu \overset{\text{LMCT}}{=} \int_E f \, d\mu}$

$(\chi_E \cdot s_n)$ is a seq of ↑ positive MF° $\Rightarrow \lim_{n\to\infty} \chi_E(x) s_n(x) = \chi_E(x) f(x) \xRightarrow{\text{LMCT}} \boxed{\lim_{n\to\infty} \int_X \chi_E \cdot s_n \, d\mu = \int_X \lim_{n\to\infty} (\chi_E s_n)\, d\mu = \int_X \chi_E f \, d\mu}$

So $\int_E f \, d\mu = \int_X \chi_E f \, d\mu$

Prop = Let $f_n : X \to [0,\infty]$ a seq of MF°, and let: $f(x) := \sum_{n=1}^{\infty} f_n(x) = \lim_{N\to\infty} \sum_{n=1}^{N} f_n(x)$

Then: $\forall E \in \mathcal{F}$, $\boxed{\int_E f\,d\mu = \sum_{n=1}^{\infty} \int_E f_n\,d\mu} = \int_E \left(\sum_{n=1}^{\infty} f_n\right) d\mu$ for positive functions.

Proof = • claim: $\int_E (f_1+f_2)\,d\mu = \int_E f_1\,d\mu + \int_E f_2\,d\mu$

(⊛) Let $\begin{cases} 0 \le s_1 \le s_2 \le \dots \le s_n \le \dots \le f_1 \\ 0 \le t_1 \le t_2 \le \dots \le t_n \le \dots \le f_2 \end{cases}$ simple MF° ST: $\begin{cases} \lim_{n\to\infty} s_n(x) = f_1(x) & \forall x \in X \\ \lim_{n\to\infty} t_n(x) = f_2(x) & \forall x \in X \end{cases} \overset{LMCT}{\Longrightarrow} \begin{cases} \lim_{n\to\infty} \int_E s_n\,d\mu = \int_E f_1\,d\mu \\ \lim_{n\to\infty} \int_E t_n\,d\mu = \int_E f_2\,d\mu \end{cases}$

Now, (s_n+t_n) is a seq of ↗ positive MF° with: $\lim_{n\to\infty} (s_n(x)+t_n(x)) = \lim_{n\to\infty} s_n(x) + \lim_{n\to\infty} t_n(x) = f_1(x)+f_2(x)$

LMCT $\Rightarrow \int_E (f_1+f_2)\,d\mu = \lim_{n\to\infty} \int_E (s_n+t_n)\,d\mu \underset{\uparrow \text{ positive simple MF°}}{=} \lim_{n\to\infty} \left(\int_E s_n\,d\mu + \int_E t_n\,d\mu\right) = \int_E f_1\,d\mu + \int_E f_2\,d\mu$ ☑

• Induction: $\forall n \in \mathbb{N}$, $\int_E (f_1+f_2+\dots+f_n)\,d\mu = \int_E f_1\,d\mu + \dots + \int_E f_n\,d\mu$

• Let $g_N = f_1+\dots+f_N = \sum_{n=1}^{N} f_n$: (g_n) is an ↗ seq of positive MF° with: $\lim_{N\to\infty} g_N(x) = \sum_{n=1}^{\infty} f_n(x)$

LMCT $\Rightarrow \lim_{N\to\infty} \int_E g_N\,d\mu = \int_E \left(\lim_{N\to\infty} g_N\right) d\mu = \int_E \left(\sum_{n=1}^{\infty} f_n\right) d\mu$

$\lim_{N\to\infty} \int_E \left(\sum_{n=1}^{N} f_n\right) d\mu = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \int_E f_n\,d\mu\right) = \sum_{n=1}^{\infty} \int_E f_n\,d\mu$

Fatou's Lemma = $f_n : X \to [0,\infty]$ a seq of MF°

Then $\forall E \in \mathcal{F}$: $\boxed{\int_E \left(\liminf_{n\to\infty} f_n\right) d\mu \le \liminf_{n\to\infty} \int_E f_n\,d\mu}$

⚠ The inequality can be strict

Ex: $0 < \infty$

Proof = • Recall: $\liminf_{n\to\infty} f_n(x) = \sup_{k\ge 1} \inf_{i \ge k} f_i(x)$

• Let $g_k(x) = \inf_{i\ge k} f_i(x) \Rightarrow g_k$'s are nonnegative MF° and (g_n) is ↗ : $g_k(x) \le g_{k+1}(x)$ $\forall k \in \mathbb{N}$

So $\lim_{k\to\infty} g_k(x) = \sup_{k\ge 1} g_k(x) = \liminf_{n\to\infty} f_n(x) \Rightarrow \boxed{\int_E \left(\liminf_{n\to\infty} f_n\right) d\mu = \int_E \left(\lim_{k\to\infty} g_k\right) d\mu \overset{LMCT}{=} \lim_{k\to\infty} \int_E g_k\,d\mu = \liminf_{k\to\infty} \int_E g_k\,d\mu}$

↳ (⊛) the limit ∃ since ↗ seq in $\mathbb{R}$.

• But $g_k(x) = \inf_{i\ge k} f_i(x) \le f_k(x) \Rightarrow \int_E g_k\,d\mu \le \int_E f_k\,d\mu \Rightarrow \liminf_{k\to\infty} \int_E g_k\,d\mu \le \liminf_{k\to\infty} \int_E f_k\,d\mu$

$\Rightarrow \int_E \left(\lim_{k\to\infty} f_k\right) d\mu = \liminf_{k\to\infty} \int_E g_k\,d\mu \le \liminf_{k\to\infty} \int_E f_k\,d\mu$

Note = We used that: $(a_n), (b_n)$ seq in $[-\infty,\infty]$, $a_n \le b_n\ \forall n \Longrightarrow \liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n$

(⊛) $\liminf_{n\to\infty} a_n = \sup_{k\ge 1}\left(\inf_{i\ge k} a_i\right) \le \sup_{k\ge 1}\left(\inf_{i\ge k} b_i\right) = \liminf_{n\to\infty} b_n$

Prop = $f : X \to [0,\infty]$ a MF° $\Longrightarrow \begin{cases} \bullet\ \forall E \in \mathcal{F}: \boxed{\phi(E) := \int_E f\,d\mu} \text{ is a measure in } (X,\mathcal{F}) \\ \bullet\ g : X \to [0,\infty] \text{ a MF°} \Longrightarrow \boxed{\int_A g\,d\phi = \int_A g f\,d\mu}\ \forall A \in \mathcal{F} \end{cases}$

⇨ "$d\phi = f\,d\mu$" $\Rightarrow \phi(E) = \int_E f\,d\mu$

Proof = • $(E_n) \in \mathcal{F}$ disjoint, $E = \bigcup_{n=1}^{\infty} E_n \Rightarrow \chi_E(x) = \sum_{n=1}^{\infty} \chi_{E_n}(x)$

(⊛) E_n disjoint $\Rightarrow \begin{cases} x \in E \Rightarrow x \in E_{n_0} \Rightarrow \chi_E(x) = 1 = \chi_{E_{n_0}}(x) = \sum_{n=1}^{\infty} \chi_{E_n}(x) \\ x \notin E \Rightarrow x \notin E_n\ \forall n \Rightarrow \chi_E(x) = 0 = \sum_{n=1}^{\infty} \chi_{E_n}(x) \end{cases}$

So $f(x)\chi_E(x) = \sum_{n=1}^{\infty} f_n(x)\chi_{E_n}(x) \Rightarrow \phi(E) = \int_E f\,d\mu \underset{\text{Prop 1}}{=} \int_X f\chi_E\,d\mu = \int_X \left(\sum_{n=1}^{\infty} f\cdot\chi_{E_n}\right) d\mu \underset{\text{Prop 2}}{=} \sum_{n=1}^{\infty} \int_X (f\cdot\chi_{E_n})\,d\mu = \sum_{n=1}^{\infty} \int_{E_n} f\,d\mu = \sum_{n=1}^{\infty} \phi(E_n)$

So $\phi(E) = \sum_{n=1}^{\infty} \phi(E_n)$ ✓ (Note: $\phi(\emptyset) = \int_\emptyset f\,d\mu = 0$)

• Goal: $\boxed{\int_A g\,d\phi = \int_A gf\,d\mu}$

linearity

Case 1 = $g = \chi_E \Rightarrow \int_A g\,d\phi = \int_A \chi_E\,d\phi = \int_X \chi_A \chi_E\,d\phi = \int_X \chi_{A\cap E}\,d\phi = \int_{A\cap E} d\phi = \phi(A\cap E) \overset{\text{def of } \phi}{=} \int_{A\cap E} f\,d\mu = \int_X \chi_{A\cap E} f\,d\mu = \int_X \chi_E \chi_A f\,d\mu = \int_A \chi_E f\,d\mu$ $\boxed{= \int_A gf\,d\mu}$

Case 2 = $g = \sum_{k=1}^{m} \gamma_k \chi_{E_k}$ a simple MF° $\Longrightarrow \int_A g\,d\phi = \sum_{k=1}^{m} \gamma_k \int_A \chi_{E_k}\,d\phi = \sum_{k=1}^{m} \gamma_k \int_A \chi_{E_k} f\,d\mu = \int_A \left(\sum_{k=1}^{m} \gamma_k \chi_{E_k}\right) f\,d\mu = \int_A gf\,d\mu$

LMCT

Case 3 = $g : X \to [0,\infty]$ a general MF°: take a seq (s_n) of ↗ simple MF° $0 \le s_1 \le s_2 \le \dots \le s_n \le \dots \le g$, ST $s_n \to g$ as $n\to\infty$

by Case 2: $\int_A s_n\,d\phi = \int_A s_n f\,d\mu \underset{LMCT}{\Longrightarrow} \int_A g\,d\phi = \int_A \lim_{n\to\infty} s_n\,d\phi = \lim_{n\to\infty} \int_A s_n\,d\phi = \lim_{n\to\infty} \int_A s_n f\,d\mu \underset{LMCT}{=} \int_A gf\,d\mu$

Integration of Complex functions :

I) Integrability over $\mathbb{C}$:

<u>Recall</u> = $(X, \mathcal{F}, \mu)$ a MSpace, $f : X \longrightarrow \mathbb{C}$ where $f = u + iv$ for $u, v : X \longrightarrow \mathbb{R}$

f is measurable $\Longleftrightarrow$ u & v are measurable ; f measurable $\Longrightarrow |f| = \sqrt{u^2+v^2}$ is measurable

<u>Integrability</u> = a MF $f : X \longrightarrow \mathbb{C}$ is integrable over $E \in \mathcal{F}$ if . $\int_E |f| \, d\mu < \infty$

<u>Note</u> : if $E = X$, we say f is integrable

<u>Prop</u> = $f : X \longrightarrow \mathbb{C}$ is integrable over $E \in \mathcal{F}$ $\Longleftrightarrow$ $u, v : X \to \mathbb{R}$ are integrable over E $(f = u + iv)$

<u>Proof</u> : $\boxed{\Longrightarrow}$ f integrable : $|f| \geq |u|, |v| \Rightarrow \int_E |u| \, d\mu , \int_E |v| \, d\mu \leq \int_E |f| \, d\mu < \infty$

$\boxed{\Longleftarrow}$ u, v integrable : $|f| = \sqrt{u^2+v^2} \leq |u| + |v| \rightarrow \int_E |f| \, d\mu \leq \int_E |u| \, d\mu + \int_E |v| \, d\mu < \infty$

<u>Note</u> : The set of all integrable functions over $E \in \mathcal{F}$ forms a <u>Vspace</u> !

(*) $|\alpha f + \beta g| \leq |\alpha| \cdot |f| + |\beta| \cdot |g| < \infty$

<u>f_+ , f_-</u> : $f : X \longrightarrow \mathbb{R}$ a <u>real</u> function : $\begin{cases} f_+(x) := \max(f(x), 0) \geq 0 \\ f_-(x) := -\min(f(x), 0) \geq 0 \end{cases}$ both are MF $\Longrightarrow$ $\boxed{f(x) = \begin{cases} f_+(x) & , \text{if } f(x) \geq 0 \\ 0 & , \text{if } f(x) = 0 \\ -f_-(x) & , \text{if } f(x) \leq 0 \end{cases}}$

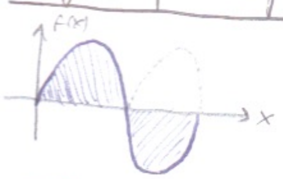
<u>Prop</u> = $f : X \to \mathbb{R}$ integrable over $E \in \mathcal{F} \Longrightarrow f_+ , f_-$ integrable over E.

(*) $|f(x)| = f_+(x) + f_-(x) \geq f_+(x), f_-(x) \Rightarrow \int_E f_+ \, d\mu , \int_E f_- \, d\mu \leq \int_E |f| \, d\mu < \infty$.

$\boxed{\begin{cases} f(x) = f_+(x) - f_-(x) \\ |f(x)| = f_+(x) + f_-(x) \end{cases}}$

<u>Integral of a Real function</u> : $f : X \to \mathbb{R}$ integrable over $E \in \mathcal{F}$: $\boxed{\int_E f \, d\mu := \int_E f_+ \, d\mu - \int_E f_- \, d\mu}$

<u>Integral of a Complex function</u> : $f : X \to \mathbb{C}$ integrable over $E \in \mathcal{F}$:



$$\boxed{\int_E f \, d\mu := \int_E u \, d\mu + i \int_E v \, d\mu} = \left(\int_E u_+ \, d\mu - \int_E u_- \, d\mu \right) + i \left(\int_E v_+ \, d\mu - \int_E v_- \, d\mu \right)$$

II) Properties of the Integral :

<u>Prop</u> : $f, g : X \longrightarrow \mathbb{C}$ integrable functions over $E \in \mathcal{F}$:

(1) $\int_E (f+g) \, d\mu = \int_E f \, d\mu + \int_E g \, d\mu$

(2) $\int_E (\alpha f) \, d\mu = \alpha \int_E f \, d\mu \quad , \alpha \in \mathbb{C}$

$\Longrightarrow \boxed{\int_E (\alpha f + \beta g) \, d\mu = \alpha \int_E f \, d\mu + \beta \int_E g \, d\mu} \quad \forall \alpha, \beta \in \mathbb{C}$

(3) $\left| \int_E f \, d\mu \right| \leq \int_E |f| \, d\mu$

Proof =

① $\int_E (f+g)\,d\mu = \int_E f\,d\mu + \int_E g\,d\mu$:

Proof = $\begin{cases} f = u_1 + iv_1 \\ g = u_2 + iv_2 \end{cases} \Rightarrow \cdot\ f+g = (u_1+u_2) + i(v_1+v_2) \rightarrow \int_E (f+g)\,d\mu = \int_E (u_1+u_2)\,d\mu + i\int_E (v_1+v_2)\,d\mu$.

$\cdot \begin{cases} \int_E f\,d\mu = \int_E u_1\,d\mu + i\int v_1\,d\mu \\ \int_E g\,d\mu = \int_E u_2\,d\mu + i\int v_2\,d\mu \end{cases} \Rightarrow \int_E f\,d\mu + \int_E g\,d\mu = \left(\int_E u_1\,d\mu + \int_E u_2\,d\mu\right) + i\left(\int_E v_1\,d\mu + \int_E v_2\,d\mu\right)$

$\Rightarrow$ **Need** : $\begin{cases} \int_E (u_1+u_2)\,d\mu = \int_E u_1\,d\mu + \int_E u_2\,d\mu \\ \int_E (v_1+v_2)\,d\mu = \int_E v_1\,d\mu + \int_E v_2\,d\mu \end{cases}$

Lemma : $f, g : X \to \mathbb{R}$ real : $\int_E (f+g)\,d\mu = \int_E f\,d\mu + \int_E g\,d\mu$

(p) let $h = f+g$: $\begin{cases} h = h_+ - h_- \\ f = f_+ - f_- \\ g = g_+ - g_- \end{cases} \Rightarrow h_+ + f_- + g_- = h_- + f_+ + g_+$ are all positive functions (we proved linearity for positive f's)

So $\int_E (h_+ + f_- + g_-)\,d\mu = \int_E (h_- + f_+ + g_+)\,d\mu \Rightarrow \int_E h_+\,d\mu + \int_E f_-\,d\mu + \int_E g_-\,d\mu = \int_E h_-\,d\mu + \int_E f_+\,d\mu + \int_E g_+\,d\mu$

So $\left(\int_E h_+\,d\mu - \int_E h_-\,d\mu\right) = \left(\int_E f_+\,d\mu - \int_E f_-\,d\mu\right) + \left(\int_E g_+\,d\mu - \int_E g_-\,d\mu\right) \Rightarrow \int_E h\,d\mu = \int_E f\,d\mu + \int_E g\,d\mu$ ☑

② $\int_E (\alpha f)\,d\mu = \alpha \int_E f\,d\mu$:

Proof = • Case 1 = $\alpha \in \mathbb{R}$, $f : X \to \mathbb{R}$ is real =

• if $\alpha \geq 0$: $(\alpha f_+) = \max(\alpha f, 0) = \alpha \max(f, 0) = \alpha f_+ \rightarrow \begin{cases} (\alpha f)_+ = \alpha \cdot f_+ \\ (\alpha f)_- = \alpha \cdot f_- \end{cases}$

So $\int_E (\alpha f)\,d\mu = \int_E (\alpha f)_+\,d\mu - \int (\alpha f)_-\,d\mu = \int_E (\alpha f_+)\,d\mu - \int_E (\alpha f_-)\,d\mu = \alpha \int_E f_+\,d\mu - \alpha \int_E f_-\,d\mu = \alpha \int_E f\,d\mu$

• if $\alpha = -1$: $(-f)_+ = \max(-f, 0) = -\min(f, 0) = f_- \Rightarrow \begin{cases} (-f)_+ = f_- \\ (-f)_- = f_+ \end{cases}$

So $\int_E (-f)\,d\mu = \int_E (-f)_+\,d\mu - \int_E (-f)_-\,d\mu = \int_E f_-\,d\mu - \int_E f_+\,d\mu = -\int_E f\,d\mu$

• if $\alpha < 0$: $\alpha = -|\alpha|$ $\int_E (\alpha f)\,d\mu = \int_E (-|\alpha| f)\,d\mu = -\int_E |\alpha| f\,d\mu = -|\alpha| \int_E f\,d\mu = \alpha \int_E f\,d\mu$

• Case 2 = $\alpha \in \mathbb{R}$, $f : X \to \mathbb{C}$ is complex =

So $\alpha f = \underbrace{\alpha u}_{\text{real}} + i\underbrace{\alpha v}_{\text{real}} \Rightarrow \int_E \alpha f\,d\mu = \int_E \underset{\in \mathbb{R}}{\alpha} u\,d\mu + i\int_E \underset{\in \mathbb{R}}{\alpha} v\,d\mu = \alpha \int_E u\,d\mu + i\alpha \int_E v\,d\mu = \alpha \int_E f\,d\mu$.

• Case 3 = $\alpha = i$, $f : X \to \mathbb{C}$ is complex =

So $i \cdot f = iu - v \Rightarrow \int_E if\,d\mu = \int_E (\underset{\in \mathbb{R}}{-v})\,d\mu + i\int_E u\,d\mu = -\int_E v\,d\mu + i\int_E u\,d\mu = i\int_E f\,d\mu$

• Case 4 = $\alpha = a + ib$, $f : X \to \mathbb{C}$ complex =

$\int_E \alpha f\,d\mu = \int_E (af + ibf)\,d\mu \underset{\text{prev prop}}{=} \int_E (af)\,d\mu + \int_E ibf\,d\mu = a\int_E f\,d\mu + b\int_E if\,d\mu = a\int_E f\,d\mu + ib\int_E f\,d\mu = \alpha \int_E f\,d\mu$

③ $\left|\int_E f\,d\mu\right| \leq \int_E |f|\,d\mu$:

Proof = • Case 1 = $f : X \to \mathbb{R}$ is real :

So $\begin{cases} f = \overset{\geq 0}{f_+} - \overset{\geq 0}{f_-} \\ |f| = f_+ + f_- \end{cases} \Rightarrow \left|\int_E f\,d\mu\right| = \left|\int_E (f_+ - f_-)\,d\mu\right| = \left|\int_E f_+\,d\mu - \int_E f_-\,d\mu\right| \leq \left|\int_E f_+\,d\mu\right| + \left|\int_E f_-\,d\mu\right| \overset{f_+, f_- \geq 0}{=} \int_E f_+\,d\mu + \int_E f_-\,d\mu$

$= \int_E (f_+ + f_-)\,d\mu = \int_E |f|\,d\mu$

• Case 2 = $f : X \to \mathbb{C}$ is complex :

$z := \int_E f\,d\mu \Rightarrow z \in \mathbb{C}$: $z = |z| e^{i\alpha}$ (polar form) $\Rightarrow |z| = z e^{-i\alpha} \overset{\text{def of } z}{=} e^{-i\alpha} \int_E f\,d\mu = \int_E e^{-i\alpha} \cdot f\,d\mu$

So $\underset{0 \leq}{\left|\int_E f\,d\mu\right| = |z|} = \int_E \underbrace{e^{-i\alpha} f}_{= u + iv}\,d\mu = \int_E (u+iv)\,d\mu = \int_E u\,d\mu + i\underbrace{\int_E v\,d\mu}_{= 0} = \int_E u\,d\mu \leq \int_E |u|\,d\mu \leq \int |f|\,d\mu$

($|z|$ has no imaginary part) $\qquad |f| = |e^{-i\alpha} f| = |u + iv| = \sqrt{u^2 + v^2} \geq |u|$

III Lebesgue Dominated Convergence Theorem :

LDCT = $(f_n)_{n=1}^{\infty}$ MF°, $f_n : X \longrightarrow \mathbb{C}$, $E \in \mathfrak{M}$:

suppose : • $\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in E$

• $\exists\, g : X \longrightarrow [0,\infty)$ ST $\begin{cases} |f_n(x)| \le g(x) : \forall n \in \mathbb{N}, \forall x \in E \\ \int_E g\, d\mu < \infty \end{cases}$

then : $\begin{cases} \lim_{n\to\infty} \int_E |f - f_n|\, d\mu = 0 \\ \lim_{n\to\infty} \int_E f_n\, d\mu = \int_E f\, d\mu \end{cases}$

Note = the fact that f is integrable is implied by the statement

Proof = Set $\begin{cases} f_n(x) = 0 \\ g(x) = 0 \end{cases} \forall x \notin E$ and take $E = X$ (otherwise replace f_n by $f_n \chi_E$)

• f is measurable = (∵) f is a limit of MF°

• f is integrable = (∵) $|f_n(x)| \le g(x) \Rightarrow f_n$ is integrable $\forall n \Rightarrow |f(x)| = \lim_{n\to\infty} |f_n(x)| \le g(x) \Rightarrow f$ is integrable.

• Set $h_n := 2g - |f - f_n| \geqslant 0$ (∵) $|f - f_n| \le |f| + |f_n| \le |g| + |g| = 2g$. as $g \ge 0$.

Fatou's Lemma $\Longrightarrow \int_X \liminf_{n\to\infty} h_n\, d\mu \le \liminf_{n\to\infty} \int_X h_n\, d\mu$

does not matter, if it's X or E

So : • $\liminf_{n\to\infty} h_n = \liminf_{n\to\infty} (2g - |f - f_n|) = 2g + \underbrace{\liminf (-|f - f_n|)}_{=0 \text{ as } \lim_{n\to\infty} f_n = f} = 2g$ (at fixed x)

• $\liminf_{n\to\infty} \int_X h_n\, d\mu = \liminf_{n\to\infty} \left(\int_X 2g\, d\mu - \int_X |f - f_n|\, d\mu \right) = \int_X 2g\, d\mu + \liminf_{n\to\infty} \left(-\int_X |f - f_n|\, d\mu \right) = \int_X 2g\, d\mu - \limsup_{n\to\infty} \int_X |f - f_n|\, d\mu$

Fatou $\Longrightarrow \int_X 2g\, d\mu \le \int_X 2g\, d\mu - \limsup_{n\to\infty} \int_X |f - f_n|\, d\mu \Rightarrow \limsup_{n\to\infty} \underbrace{\int_X |f - f_n|\, d\mu}_{\geqslant 0} \le 0$

But this is a seq of positive real nb $\Rightarrow \limsup_{n\to\infty} \int_X |f - f_n|\, d\mu = 0$.

Also : $\liminf_{n\to\infty} \underbrace{\int_X |f - f_n|\, d\mu}_{\geqslant 0} = 0$ (∵) $0 \geqslant \limsup \geqslant \liminf \geqslant 0$. $\Big\} \Rightarrow \limsup = \liminf = 0$.

$\Longrightarrow \boxed{\lim_{n\to\infty} \int_X |f - f_n|\, d\mu = 0}$

• $\left| \int_X f\, d\mu - \int_X f_n\, d\mu \right| = \left| \int_X (f - f_n)\, d\mu \right| \le \int_X |f - f_n|\, d\mu \longrightarrow 0$ as $n \to \infty$

$\Rightarrow \lim_{n\to\infty} \left| \int_X f\, d\mu - \int_X f_n\, d\mu \right| = 0 \Rightarrow \lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f\, d\mu$

μ-almost Everywhere :

(I) Definition & properties

μ-almost everywhere / μ-almost all x = $(X, \mathcal{F}, \mu)$ a MSp , $E \in \mathcal{F}$, $x \in E$

A mathematical statement $S(x)$ holds μ-AE / μ-AA $x \in E$ if:

$\exists F \in \mathcal{F}$, $F \subset E$ and $\mu(F) = 0$ ST $\{x \in E : S(x) \text{ does not hold}\} \subset F$.

Note = $\{x \in E : S(x) \text{ does } \underline{\underline{\text{NOT}}} \text{ hold}\}$ does NOT need to be a MSet.

⚠ Sets of measure zero don't matter ... Unless take an uncountable union of them !

Prop : $f : X \longrightarrow [0, \infty]$ a MF°, $E \in \mathcal{F}$, $\mu(E) > 0$:

Then: $\int_E f \, d\mu = 0 \implies f(x) = 0$ μ-AA $x \in E$.

Proof = $S(x)$ = "we have $f(x) = 0$" $\longrightarrow F := \{x \in E : S(x) \text{ holds}\} = \{x \in E : f(x) > 0\} \implies F$ is a Mset !

Goal : $\boxed{\mu(F) = 0}$

(S) $E_n := \{x \in E : f(x) \geq \frac{1}{n}\} \implies \int_E f \, d\mu \geq \int_{E_n} f \, d\mu$ as $E_n \subseteq E$

So $0 = \int_E f \, d\mu \geq \int_{E_n} f \, d\mu \geq \int_{E_n} \frac{1}{n} \, d\mu = \frac{1}{n} \int_{E_n} d\mu = \frac{1}{n} \mu(E_n) \implies \mu(E_n) = 0 \; \forall n$

$\implies \mu(F) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n) = 0$

Prop = $f : X \longrightarrow \mathbb{C}$ an integrable function :

$\int_E f \, d\mu = 0 \quad \forall E \in \mathcal{F} \implies f(x) = 0$ μ-AA x

Proof = $f = u + iv$, $E_1 = \{x \in X : u_+(x) > 0\} \implies 0 = \int_{E_1} f \, d\mu = \left(\int_{E_1} u_+ \, d\mu - \underbrace{\int_{E_1} u_- \, d\mu}_{=0}\right) + \underbrace{i \int_{E_1} v \, d\mu}_{=0}$

Im = 0 since $\int_{E_1} f \, d\mu = 0$

$\implies \int_{E_1} u_+ \, d\mu = 0$, $\mu(E_1) > 0 \overset{\text{Prev Prop}}{\implies} u_+(x) \equiv 0$ μ-AE on E_1 $\boxed{\implies \Leftarrow}$

$\hookrightarrow u_+ > 0$ on E_1

So $\mu(E_1) = 0$

Similarly : $\left.\begin{cases} E_2 = \{x \in X : u_-(x) > 0\} \implies \mu(E_2) = 0 \\ E_3 = \{x \in X : v_+(x) > 0\} \implies \mu(E_3) = 0 \\ E_4 = \{x \in X : v_-(x) > 0\} \implies \mu(E_4) = 0 \end{cases}\right\} \implies \mu(\{x : f(x) \neq 0\}) = \mu(E_1 \cup E_2 \cup E_3 \cup E_4)$

$\leq \mu(E_1) + \mu(E_2) + \mu(E_3) + \mu(E_4) = 0$

Prop = $f : X \longrightarrow [0, \infty]$ a MF° : f integrable $\implies f(x) < \infty$ μ-AA $x \in X$

Proof = Let $F := \{x : f(x) = \infty\} \rightsquigarrow$ M Set ! Goal : $\mu(F) = 0$ $\rightsquigarrow$ USE : f int $\implies \int_X f \, d\mu < \infty$

[?] $F \subset X$, $f \geq 0$ $\implies \int_F f \, d\mu \leq \int_X f \, d\mu < \infty \implies \infty > \int_F f \, d\mu = \int_F \infty \cdot d\mu = \infty \cdot \int_F d\mu = \infty \cdot \mu(F) \implies \mu(F) = 0$

OR : $F_n = \{x \in X : f(x) \geq n\}$: $\infty > \int_X f \, d\mu \geq \int_{F_n} f \, d\mu \geq \int_{F_n} n \, d\mu = n\mu(F_n) \geq 0 \implies \mu(F_n) \leq \frac{1}{n} \int_X f \, d\mu \xrightarrow[n \to \infty]{} 0 \implies \mu(F) = \lim_{n \to \infty} \mu(F_n) = 0$

Prop = $(X, \mathcal{F}, \mu)$ a MSp , $\{E_n\}_{n=1}^{\infty}$ a Seq of MSets : $\sum_{n=1}^{\infty} \mu(E_n) < \infty \implies \mu$AA $x \in X$ lie in at most finitely many sets E_n.

Proof 1 = $F = \{x \in X : x \text{ belongs to } \underline{\infty} \text{ many sets } E_n\} \implies F = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \rightsquigarrow$ Goal : $\mu(F) = 0$

$\mu(F) \leq \mu\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \sum_{k=n}^{\infty} \mu(E_k) \quad \forall n \in \mathbb{N}$: but $\sum_{k=1}^{\infty} \mu(E_k) < \infty \implies \lim_{n \to \infty} \sum_{k=n}^{\infty} \mu(E_k) = 0 \geq \mu(F) \geq 0 \implies \mu(F) = 0$

Proof 2 : Set $f(x) = \sum_{n=1}^{\infty} \chi_{E_n}(x)$: $\{x : f(x) = \infty\} = \{x : x \text{ belongs to } \infty \text{ many sets } E_n\}$.

f integrable : $(\because)$ $\int_X f \, d\mu = \int_X \left(\sum_{n=1}^{\infty} \chi_{E_n}(x)\right) d\mu = \sum_{n=1}^{\infty} \int_X \chi_{E_n} \, d\mu = \sum_{n=1}^{\infty} \mu(E_n) < \infty$

Prev prop $\implies \mu(\{x : f(x) = \infty\}) = 0$ ✓

Borel 0-1 law = $(X, \mathcal{F}, \mu)$ a prob. space, $\{E_n\}_{n=1}^{\infty}$ a seq of indep events in $\mathcal{F}$

$$\sum_{n=1}^{\infty} \mu(E_n) = \infty \implies \mu\text{-AA } x \in X \text{ lie in } \infty \text{ many of the } E_n.$$

RK: • $F := \{x \in X : x \text{ lies in } \infty \text{ many } E_n\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \implies \begin{cases} \sum_{n=1}^{\infty} \mu(E_n) < \infty \xrightarrow{\text{Borel-Cantelli}} \mu(F) = 0 \\ \sum_{n=1}^{\infty} \mu(E_n) = \infty \xrightarrow[\text{NEED indep!}]{\text{Borel 0-1 law}} \mu(F) = 1 \end{cases}$ → does not require indep

• Borel 0-1 law fails if $\{E_n\}_{n=1}^{\infty}$ are not indep

Proof = $F = \{x \in X : x \text{ lies in } \infty \text{ many } E_n\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$ → Goal: $\mu(F) = 1$

• claim: Let $F_n = \bigcup_{k=n}^{\infty} E_k$: $\mu(F) = 1 \iff \mu(F_n) = 1 \ \forall n$ (pf) $\boxed{\Rightarrow}$ $F = \bigcap_{n=1}^{\infty} F_n$: $1 = \mu(F) \leq \mu(F_n) \leq 1 \implies \mu(F_n) = 1 \ \forall n$

• claim: $\mu(F_n) = 1 \ \forall n \in \mathbb{N}$ (so $\mu(F) = 1$) $\boxed{\Leftarrow}$ $\mu(F_n) = 1 \ \forall n \Rightarrow F^c = \bigcup_{n=1}^{\infty} F_n^c \Rightarrow \mu(F^c) \leq \underbrace{\sum_{n=1}^{\infty} \mu(F_n^c)}_{=0} = 0$

$\Rightarrow \mu(F) = 1 - \mu(F^c) = 1$.

(pf) Note: $e^{-x} \geq 1 - x \ \forall x \geq 0$

E_k indep $\Rightarrow E_k^c$ indep (Hwe)

So $e^{-\sum_{k=n}^{\infty} \mu(E_k)} = \prod_{k=n}^{\infty} e^{-\mu(E_k)} \geq \prod_{k=n}^{\infty} (1 - \mu(E_k)) = \prod_{k=n}^{\infty} \mu(E_k^c) \geq \mu\left(\bigcap_{k=n}^{\infty} E_k^c\right) = 1 - \mu\left(\bigcup_{k=n}^{\infty} E_k\right)$

$\sum_{k=1}^{\infty} \mu(E_k) = \infty \implies 0 = \lim_{N \to \infty} e^{-\sum_{k=n}^{N} \mu(E_k)} \geq 1 - \lim_{N \to \infty} \mu\left(\bigcup_{k=n}^{N} E_k\right) = 1 - \underbrace{\mu\left(\bigcup_{k=n}^{\infty} E_k\right)}_{\leq 1 \text{ (prob space)}} \geq 0 \implies \mu\left(\bigcup_{k=n}^{\infty} E_k\right) = 1$ ✓

II Measurable Transformations & Induced Measures:

Measurable Transfo (MT°) = $(X, \mathcal{F})$ and $(Y, \mathcal{T})$ Msp: $T : X \to Y$ is a MT° if $\boxed{T^{-1}(E) \in \mathcal{F}, \ \forall E \in \mathcal{T}}$

Induced measure = Let μ a measure on $(X, \mathcal{F})$: $\mu_T(E) := \mu(T^{-1}(E))$ is the measure induced by the MT° T.

Prop = μ_T is a measure on $(Y, \mathcal{T})$

(pf) Let $\{E_n\}_{n=1}^{\infty}$ a seq of disjoint elements of $\mathcal{T}$: then $\mu_T\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(T^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right)\right) \overset{\text{disjoint union}}{=} \mu\left(\bigcup_{n=1}^{\infty} T^{-1}(E_n)\right) = \underbrace{\sum_{n=1}^{\infty} \mu(T^{-1}(E_n))}_{\text{disjoint in } \mathcal{F}} = \sum_{n=1}^{\infty} \mu_T(E_n)$

Distribution Measure of f = Let $(Y, \mathcal{T}) = (\mathbb{R}, \mathcal{B})$ → Borel σ-Alg in $\mathbb{R}$: min σ-Alg containing all open sets. , $f : X \to \mathbb{R}$ a MF°

the measure μ_f induced by f on $(\mathbb{R}, \mathcal{B})$ is the "distribution measure of f"

Note = $f : X \to \mathbb{R}$ a MF° $\Rightarrow f : X \to \mathbb{R}$ a MT° (pf) f is a MF° $\Rightarrow f^{-1}(E) \in \mathcal{F} \ \forall E \in \mathcal{B} \Rightarrow f$ is a MT°

Lemma = $g : \mathbb{R} \to \mathbb{R}$ a Borel MF° : $g \circ f : X \to \mathbb{R}$ is also a MF°

(pf) V open in $\mathbb{R}$: $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \overset{g \text{ MF°}}{=} f^{-1}(\text{set in } \mathcal{B}) \in \mathcal{F}$ as $f^{-1}(\text{Borel})$ are M sets.

Prop = $f : X \to \mathbb{R}$ a MF°, $g : \mathbb{R} \to [0, \infty)$ a Borel MF° : then $\boxed{\int_X (g \circ f) \, d\mu = \int_{\mathbb{R}} g \, d\mu_f}$

Proof: • Case 1 : $g = \chi_E$ for some $E \in \mathcal{B}$.

$(g \circ f)(x) = \chi_E(f(x)) = \begin{cases} 1 ; & f(x) \in E \\ 0 ; & f(x) \notin E \end{cases} = \chi_{f^{-1}(E)}(x) \implies \int_X g \circ f \, d\mu = \int_X \chi_{f^{-1}(E)} \, d\mu = \mu(f^{-1}(E)) = \mu_f(E) = \int_{\mathbb{R}} \overset{g}{\widetilde{\chi_E}} \, d\mu_f$

• Case 2 = $g = \sum_{k=1}^{n} Y_k \chi_{E_k}$ → linearity of integral $\Rightarrow \int_X (g \circ f) \, d\mu = \int_{\mathbb{R}} g \, d\mu_f$

• Case 3 = General $g : \mathbb{R} \to [0, \infty)$ → choose simple f° $0 \leq S_1 \leq S_2 \leq \dots \leq S_n \leq \dots \leq g$ ST $\lim_{n \to \infty} S_n(x) = g(x) \ \forall x \in \mathbb{R}$

then $\begin{cases} S_n(f(x)) \geq 0 \ \forall n \text{ is increasing in } n \\ \lim_{n \to \infty} S_n(f(x)) = g(f(x)) \ \forall x \in X \end{cases} \implies \int_X (S_n \circ f) \, d\mu = \int_{\mathbb{R}} S_n \, d\mu_f \xrightarrow[n \to \infty]{\text{LMCT (both sides)}} \int_X (g \circ f) \, d\mu = \int_{\mathbb{R}} g \, d\mu_f$.

Prop = $f : X \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ MF° : then $\boxed{\int_X (g \circ f) \, d\mu = \int_{\mathbb{R}} g \, d\mu_f}$ when g integrable on $(\mathbb{R}, \mathcal{B}, \mu_f)$

Proof = $g = g_+ - g_-$ where $\begin{cases} g_+(x) = \max(0, g(x)) \\ g_-(x) = -\min(0, g(x)) \end{cases} \implies \begin{cases} (g \circ f)_+ = \max(0, g(f(x))) = g_+(f(x)) = g_+ \circ f \\ (g \circ f)_- = -\min(0, g(f(x))) = g_-(f(x)) = g_- \circ f \end{cases}$

So $\begin{cases} \int_X (g \circ f)_+ \, d\mu = \int_{\mathbb{R}} g_+ \, d\mu_f \\ \int_X (g \circ f)_- \, d\mu = \int_{\mathbb{R}} g_- \, d\mu_f \end{cases} \implies \int_X (g \circ f) \, d\mu = \int_{\mathbb{R}} g \, d\mu_f$

g int on $(\mathbb{R}, \mathcal{B}, \mu_f)$
$\iff g \circ f$ int on $(X, \mathcal{F}, \mu)$

III Link with probability :

Random Variable : $(X, \mathcal{F}, \mu)$ a prob space : a random Var is a MF° $f : X \longrightarrow \mathbb{R}$

Expectation Value : when f is integrable : $\boxed{\mathbb{E}(f) := \int_X f \, d\mu = \int_{\mathbb{R}} x \, d\mu_f}$ (⊛) $g \circ f = x \circ f$ when $g(x) = x$, $g : \mathbb{R} \to \mathbb{R}$

Probability distribution of the r.v. f : μ_f $\longrightarrow$ All you can learn about f is contained in μ_f

Prop : • $\mu_f(\mathbb{R}) = 1$ (⊛) $\mu_f(\mathbb{R}) = \int_{\mathbb{R}} d\mu_f = \int_X d\mu = \mu(X) = 1$

• $\mathbb{E}(g \circ f) = \int_{\mathbb{R}} g \, d\mu_f$ (⊛) $\mathbb{E}(g \circ f) = \int_X (g \circ f) \, d\mu = \int_{\mathbb{R}} g \, d\mu_f$

Ex : • $f \sim N(m, \sigma^2)$, $\sigma > 0$: $\mu_f([a,b]) = \frac{1}{\sigma \sqrt{2\pi}} \int_a^b e^{-\frac{(x-m)^2}{2\sigma^2}} dx$ $\longrightarrow$ Gaussian R.V.

• $f \sim \mathrm{Poiss}(\lambda)$, $\lambda > 0$: $\mu_f(E) = \sum_{\substack{k \in \mathbb{N} \cup \{0\} \\ k \in E}} e^{-\lambda} \cdot \frac{\lambda^k}{n!}$ $\longrightarrow$ Poisson R.V.

Distribution function of a R.V. f : $\forall x \in \mathbb{R}$: $F(x) := \mu_f((-\infty, x]) = \mu_f(\{x \in X : f(x) \in (-\infty, x]\})$

Prop : • for $a < b$: $F(b) - F(a) = \mu_f((a,b]) = \mu_f(\{x \in X : a < f(x) \le b\})$ (⊛) $F(b) - F(a) = \mu_f((-\infty, b]) - \mu_f((-\infty, a])$
$= \mu_f((-\infty, a]) + \mu_f((a,b]) - \mu_f((-\infty, a])$
$= \mu_f((a,b])$

• $\lim_{x \to \infty} F(x) = 1$ (⊛) $(x_n)_{n=1}^{\infty} \nearrow$ seq, $x_n \to \infty$, So : $\lim_{n \to \infty} \mu_f((-\infty, x_n]) = \mu_f\left(\bigcup_{n=1}^{\infty} (-\infty, x_n]\right) = \mu_f(\mathbb{R}) = 1$

• $\lim_{x \to -\infty} F(x) = 0$ (⊛) $(x_n)_{n=1}^{\infty} \searrow$ seq, $x_n \to -\infty$, So : $\lim_{n \to \infty} \mu_f((-\infty, x_n]) = \mu_f\left(\bigcap_{n=1}^{\infty} (-\infty, x_n]\right) = \mu_f(\emptyset) = 0$

• $\forall x \le y$: $F(x) \le F(y)$ (⊛) $(-\infty, x] \subset (-\infty, y] \Rightarrow \mu((-\infty, x]) \le \mu((-\infty, y])$

• $F : \mathbb{R} \to [0,1]$ is right continuous (⊛) $\underline{x \in \mathbb{R}}$: $(x_n)_{n=1}^{\infty} \searrow$, $\lim_{n \to \infty} x_n = x$, $x_n > x$ $\forall n \in \mathbb{N}$

So $F(x_n) = \mu_f((-\infty, x_n])$ but $\bigcap_{n=1}^{\infty} (-\infty, x_n] = (-\infty, x]$ ↳ decreasing seq of sets.

$\Rightarrow \lim_{n \to \infty} F(x_n) = \mu_f\left(\bigcap_{n=1}^{\infty} (-\infty, x_n]\right) = \mu_f((-\infty, x]) = F(x)$ True $\forall x \in \mathbb{R}$

$\longrightarrow$ For any r.v. f on $(X, \mathcal{F}, \mu)$:

$F(x) = \mu_f((-\infty, x])$ is $\longrightarrow$ increasing

$\longrightarrow$ right continuous

$\longrightarrow$ $F(-\infty) = 0$, $F(+\infty) = 1$

μ_f : probability measure on $(\mathbb{R}, \mathcal{B})$

Part 2:

Reisz Representation theorem & Lebesgue Measure

<u>Intro</u>=

- so far, Two examples of measures : $\begin{cases} \text{Dirac Measure} \\ \text{Counting Measures} \end{cases}$
- **2 ways to construct measures :**

→ Reisz Representation theorem : (Natural in functional analysis)

(*) Start with a positive linear functional Λ & get a measure on a σ-Algebra.

<u>Goal</u>: A function f is Riemann Integrable $\left(\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx\right)$ $\Leftrightarrow$ f is measurable & the set of discont of f is contained in a set of measure 0.

<u>Idea</u>: $(X, \mu, \mathcal{F})$ a M Space with : $\begin{cases} X = \text{set (metric space or Top Space)} \\ V = \text{Vspace of "nice" functions}, \; f: X \longrightarrow \mathbb{C} \end{cases}$ ↳ Top sense

$\mathcal{L}^1(X, d\mu) :=$ Vspace of all integrable functions on X $\left(\int_X |f|\,d\mu < \infty\right)$

<u>given</u> $\Lambda : V \; (\text{e.g. } \mathcal{L}^1(X, d\mu)) \longrightarrow \mathbb{C}$, a positive linear functional $\begin{pmatrix} f \geq 0 \Rightarrow \Lambda(f) \geq 0 \\ \Lambda(\alpha f + \beta g) = \alpha\Lambda(f) + \beta\Lambda(g) \end{pmatrix}$

<u>construct</u> :
- a σ-Algebra $\mathcal{F}$ on X containing all Borel sets
- a measure μ on $(X, \mathcal{F})$

ST : $\Lambda(f) = \int_X f\,d\mu \quad \forall f \in V$

<u>Note</u> : Λ extends to $V = \mathcal{L}^1(X, d\mu)$; if V is ____ ?, the measure is unique!

<u>Road to Lebesgue Measure</u>: ⟶ Use Riemann Integral!

- $f: \mathbb{R} \to \mathbb{C}$ a cont. funct^{n} with $f(x) = 0 \;\; \forall x \notin K$ (K Compact in $\mathbb{R}$) → V = set of all cont funct^{n} with $f = 0 \; \forall x \notin K_f$ compact
- Riemann integral $\Lambda(f) = \int_{\mathbb{R}} f(x)\,dx$ → induces the Lebesgue measure on $(\mathbb{R}, \mathcal{B})$ (<u>UNIQUE</u>!)

→ Carathéodory-Hahn Extension thm : (Important in prob. theory)

(*) Start with a right continuous function F & get a measure.

Given $F: \mathbb{R} \to \mathbb{R}$ right cont, $\exists!$ μ_F on $(\mathbb{R}, \mathcal{B})$ ST $\mu_F((a,b]) = F(b) - F(a)$ $\forall$ intervals $(a,b]$.
↳ Lebesgue measure

Preliminaries to the RRT :

• Local Compactness :

LCMS : (X,d) is a Locally Compact Metric Space if : every point $x \in X$ has a neighborhood whose closure is compact.

Ex : $X = \mathbb{R}^n$, or X finite dim, ...

Prop : X compact $\Rightarrow$ X locally compact (pf) Fix $x \in X$: then X is a neighb of x, and $cl(X) = X$ is compact

Prop : (X,d) a LCMS, $U \subset X$ open, $K \subset X$ compact

$$K \subset U \Rightarrow \exists\, V \text{ open ST : } \begin{cases} cl(V) \text{ is compact} \\ K \subset V \subset cl(V) \subset U \end{cases}$$

Proof : Let $x \in K$, and V_x = neighb of x ST $cl(V)$ is compact

- $\{V_x\}_{x \in K}$ = open cover of K $\longrightarrow$ pick a finite subcover : $V_{x_1}, \dots, V_{x_n}$
- Set $G = V_{x_1} \cup \dots \cup V_{x_n}$: so $cl(G) = cl(V_{x_1}) \cup \dots \cup cl(V_{x_n}) \Rightarrow cl(G)$ compact in X, $K \subset G$.
- if $U = X$: done ! (pf) set $V = G$
- if $U \subsetneq X$: Set $C = U^c$ (so $C \neq \phi$), and let $p \in C$

Then $p \notin K$ ($\because K \subset U$) $\Rightarrow \exists\, W_p$ open ST $K \subset W_p$ and $p \notin cl(W_p)$ (pf) Analysis III

Consider : $C \cap cl(G) \cap cl(W_p) \rightsquigarrow$ compact, and $p \notin$ this set (pf) $p \notin cl(W_p)$.

$$\text{So } \bigcap_{p \in C} \left[C \cap cl(G) \cap cl(W_p) \right] = \phi$$

Fin. intersect$^{\circ}$ prop $\Rightarrow \exists\, p_1, \dots, p_m \in C$ ST : $C \cap cl(G) \cap cl(W_{p_1}) \cap \dots \cap cl(W_{p_m}) = \phi$ (*)

Set $V = G \cap W_{p_1} \cap \dots \cap W_{p_m}$: V open, $K \subset V$ and $\underbrace{cl(V)}_{\text{compact}} \subset \underbrace{cl(G) \cap cl(W_{p_1}) \cap \dots \cap cl(W_{p_m})}_{\text{compact}}$

So (*) $\Rightarrow C \cap cl(V) = \phi$, $C = U^c \rightsquigarrow cl(V) \subset U$.

• Notations : (X,d) a LCMS

$\rightarrow$ Support : $f : X \to \mathbb{C}$: $Supp(f) = \{x \in X : f(x) \neq 0\}$

$\rightarrow$ compact support : $C_c(X) = \{ f : X \to \mathbb{C} \text{ cont} : Supp(f) \text{ is compact} \}$

Prop :

- $f \in C_c(X)$: $Supp(\alpha f) = \begin{cases} \phi &, \alpha = 0 \\ \alpha\, Supp(f) &, \alpha \neq 0 \end{cases}$

  (pf) • $\alpha = 0 \Rightarrow \{x : 0 \cdot f(x) \neq 0\} = \phi$
  • $\alpha \neq 0 \Rightarrow \{x : \alpha \cdot f(x) \neq 0\} = \{x : f(x) \neq 0\} = Supp(f)$
  • $f \in C_c(X) \Rightarrow (\alpha f) \in C_c(X)$ (pf) $\exists$ a finite subcover.

- $f, g \in C_c(X)$: $Supp(f+g) \subset Supp(f) \cup Supp(g)$

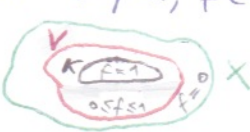
  (pf) • $\{x : f(x) + g(x) \neq 0\} \subset \{x : f(x) \neq 0\} \cup \{x : g(x) \neq 0\}$
  • $f, g \in C_c(X) \Rightarrow f + g \in C_c(X)$

$\longrightarrow$ $C_c(X)$ is a Vspace !

$\rightarrow K \prec f$: "K below f" K compact, $f \in C_c(X)$, with $\begin{cases} f(x) = 1 &, \forall x \in K \\ 0 \leq f(x) \leq 1 &, \forall x \in X \end{cases}$

$\rightarrow f \prec V$: "f below V" V open, $f \in C_c(X)$, with $\begin{cases} Supp(f) \subset V \\ 0 \leq f(x) \leq 1, \forall x \in V \end{cases}$

$\rightarrow K \prec f \prec V$:



Thm: (Urysohn's Lemma) $K \subset X$ $V \subset X$

(X,d) LCMS, K compact, V open : $K \subset V \subset X \Longrightarrow \exists f$ ST $K \prec f \prec V$

Proof: Recall : $A \subset X$, $h(x) = d(x,A) = \inf\{d(x,y) : y \in A\} \longrightarrow |h(x) - h(z)| \le d(x,z)$

A closed : $x \in A \iff d(x,A) = 0$

- Let W open ST : $cl(W)$ compact, $K \subset W \subset cl(W) \subset V$ (w/ prev prop.)
- Set $f(x) = \dfrac{d(x,W^c)}{d(x,K) + d(x,W^c)}$: f well defined, f cont, $0 \le f(x) \le 1$ (w.) $d(x,K) + d(x,W^c) = 0 \iff x \in K \cap W^c$, but $K \subset W \Rightarrow K \cap W^c = \emptyset$, so $d(x,K) + d(x,W^c) > 0$
- $f(x) = 0 \iff x \in W^c \iff x \in V^c$: (w.) $W = \{x : f(x) \neq 0\} \Rightarrow supp(f) = \underbrace{cl(W)}_{\text{compact}} \subset V$, so $supp(f)$ is compact
  $f(x) = 0 \iff d(x,W^c) = 0 \iff x \in W^c$
- $f(x) = 1 \iff d(x,K) = 0 \iff x \in K$

Note: This proof does <u>not</u> hold on Locally Compact Hausdorff spaces. $\longrightarrow$ Need a different argument
Apart from this prop, the entire proof holds for Locally Compact Hausdorff Spaces.

Prop: (X,d) a LCMS : $K \subset X$ compact, $V_1, \dots, V_n \subset X$ open

$K \subset V_1 \cup \dots \cup V_n \Longrightarrow \exists h_i$, $i = 1, \dots, n$ ST $h_i \prec V_i$ and $h_1(x) + \dots + h_n(x) = 1$ $\forall x \in K$

Note: $\{h_1, \dots, h_n\}$ = "partition of unity in K, subordinate to $\{V_1, \dots, V_n\}$"

Proof:
- Let $x \in K \subset V_1 \cup \dots \cup V_n$: then $x \in V_i$ for some $i = 1 \dots n$
- Let W_x be open ST : $cl(W_x)$ compact, and $\{x\} \subset W_x \subset cl(W_x) \subset V_i$ (w/ 1st prop)
- $\{W_x\}_{x \in K}$ = open cover of K : pick a finite subcover $\{W_{x_1}, \dots, W_{x_m}\} \Rightarrow$ By construct^{n} : $cl(W_{x_j}) \subset$ some V_i
- Given V_i, let H_i = union of all $cl(W_{x_j})$ contained in V_i : $H_i \subset V_i$, and H_i is compact (union of compacts)
- Let g_i be ST : $H_i \prec g_i \prec V_i$:

  Set $\begin{cases} h_1 = g_1 & , n = 1 \\ h_n = (1-g_1) \dots (1-g_{n-1}) g_n & , n \ge 2 \end{cases} \Longrightarrow \begin{cases} 0 \le h_i \le 1 \ \ \forall i \le n, \text{ and } \underset{\text{compact}}{supp(h_i)} \subset \underset{\text{compact}}{supp(g_i)} \\ g_i \prec V_i \Rightarrow h_i \prec V_i \\ h_1 + \dots + h_n = 1 - (1-g_1)(1-g_2) \dots (1-g_n) \end{cases}$

- We have $K \subset H_1 \cup \dots \cup H_n \subset cl(W_{x_1}) \cup \dots \cup cl(W_{x_m})$:

  so $x \in K \Rightarrow x \in H_i$ for some $i \Rightarrow g_i(x) = 1 \Rightarrow \begin{cases} (1-g_1(x)) \dots \underbrace{(1-g_i(x))}_{=0} \dots (1-g_n(x)) = 0 \\ (h_1 + \dots + h_n)(x) = 1 - 0 = 1. \end{cases}$

Reisz Representation thm :

① Theorem :

RRT : (X, d) a LCMS

$C_c(X)$ = Vspace of all continuous functions $f : X \to \mathbb{C}$ with compact support.

$\Lambda : C_c(X) \longrightarrow \mathbb{C}$: a positive linear functional $\begin{cases} \Lambda(\alpha f + \beta g) = \alpha\Lambda(f) + \beta\Lambda(g) \\ f \geq 0 \Rightarrow \Lambda(f) \geq 0 \end{cases}$

Then : $\exists$ a σ-Algebra $\mathfrak{F}$ in X that contains all Borel sets

$\exists$ a unique measure μ on $(X, \mathfrak{F})$

ST : (a) $\Lambda(f) = \int_X f \, d\mu \quad \forall f \in C_c(X)$

(b) $\mu(K) < \infty \quad \forall K$ compact

(c) $\mu(E) = \inf\{\mu(V) : E \subset V, V \text{ open}\} \quad \forall E \in \mathfrak{F}$

(d) $\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$ holds $\forall E$ open, $E \in \mathfrak{F}$ and $\mu(E) < \infty$.

(e) Let $E \in \mathfrak{F}$ and $\mu(E) = 0$: $A \subset E \Rightarrow A \in \mathfrak{F}$ $\Longrightarrow$ completeness of a measure

Construction :

- observe : • $\forall V$ open : $\mu(V) \overset{\text{attach a size to } V}{:=} \sup\{\Lambda(f) : f \prec V\}$, so : $V_1 \subset V_2 \overset{f \prec V_1 \Rightarrow f \prec V_2}{\Rightarrow} \mu(V_1) \leq \mu(V_2)$
  - $\forall E$ open : $\mu(E) := \inf\{\mu(V) : E \subset V, V \text{ open}\}$
  - $\forall E \subset X$: $\mu(E) := \inf\{\mu(V) : E \subset V, V \text{ open}\}$

$\Rightarrow$ For any $E \subset X$: $\boxed{\mu(E) := \inf\{\mu(V) : E \subset V, V \text{ open}\}}$ $\longrightarrow$ μ is countably additive on a σ-Algebra $\mathfrak{F}$ we will construct

- Define : $\mathfrak{F}_F := \{E : E \subset X, \mu(E) < \infty, \mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}\}$

$\boxed{\mathfrak{F} := \{E : E \subset X, E \cap K \in \mathfrak{F}_F \ \forall K \text{ compact}\}}$

observations :

- (c) is always True by def of μ.
- (e) holds :

  (∵) • $E_1 \subset E_2 \Rightarrow \mu(E_1) \leq \mu(E_2) \quad \forall E_1, E_2 \subset X$ $\longrightarrow$ take $\begin{cases} E_1 = A \\ E_2 = E \end{cases}$: $A \subset E \Rightarrow \mu(A) \leq \mu(E) = 0 \Rightarrow \mu(A) = 0$

  • $\mu(E) = 0 \Rightarrow E \in \mathfrak{F}$

  (∵) $\mu(E) = 0 \Rightarrow E \in \mathfrak{F}_F \Rightarrow E \cap K \in \mathfrak{F}_F \ \forall K$ compact $\Rightarrow E \in \mathfrak{F}$.

- Λ is monotone :

  $\forall f, g \in C_c(X)$: if $f(x) \geq g(x) \geq 0 \ \forall x \in X$, then : $\Lambda(f) \geq \Lambda(g)$

  (∵) $\Lambda(f) = \Lambda(f - g + g) = \underbrace{\Lambda(f-g)}_{\geq 0} + \Lambda(g) \geq \Lambda(g)$

  (∵) $(f-g) \in C_c(X)$: $f - g \geq 0 \Rightarrow \Lambda(f-g) \geq 0$

(IV) Proof:

Uniqueness:

(e) not required

Suppose (a) to (d) holds for two measures μ_1 & μ_2.

Goal: $\mu_1(K) = \mu_2(K)$ $\forall K$ compact, i.e. μ_1 & μ_2 are uniquely determined by its values on compact sets.

(PS) Let K be compact: (b) $\Rightarrow \mu(K_1) < \infty$, $\mu(K_2) < \infty$

Let $\varepsilon > 0$, Let V open ST: $K \subset V$ and $\mu_2(V) < \mu_2(K) + \varepsilon$

Urysohn's Lemma $\Rightarrow \exists f$ ST: $K \prec f \prec V$ $\longrightarrow$ so $\chi_K \leq f \leq \chi_V$

$$\Rightarrow \mu_1(K) = \int_X \chi_K \, d\mu_1 \underset{\chi_K \leq f}{\leq} \int_X f \, d\mu_1 = \Lambda(f) = \int_X f \, d\mu_2 \underset{f \leq \chi_V}{\leq} \int_X \chi_V \, d\mu_2 = \mu_2(V) \leq \mu_2(K) + \varepsilon$$

So given $\varepsilon > 0$: $\mu_1(K) \leq \mu_2(K) + \varepsilon$ $\Longrightarrow$ $\mu_1(K) \leq \mu_2(K)$ and $\mu_2(K) \leq \mu_1(K)$ (interchange the roles of μ_1 & μ_2)

$\Longrightarrow \mu_1(K) = \mu_2(K)$ $\forall K$ compact

Existence: 10 steps:

1] $(E_n)_{n=1}^{\infty} \in X$: $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n)$

"Lower Riemann Sum"

2] K compact: $\mu(K) < \infty$ and $K \in \mathfrak{F}_F$, and $\mu(K) = \inf\{\Lambda(f) : K \prec f\}$

3] V open: $\mu(V) = \sup\{\mu(K) : K \subset V, K \text{ compact}\}$

Note: $\mu(V) < \infty \Rightarrow V \in \mathfrak{F}_F$

4] $(E_n)_{n=1}^{\infty} \in \mathfrak{F}_F$ disjoint: • $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$

• $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) < \infty \Rightarrow \left(\bigcup_{n=1}^{\infty} E_n\right) \in \mathfrak{F}_F$

5] $E \in \mathfrak{F}_F$, $\varepsilon > 0$ $\Rightarrow \exists V$ open, $\exists K$ compact ST: $K \subset E \subset V$ and $\mu(V \setminus K) < \varepsilon$

6] $A, B \in \mathfrak{F}_F \Rightarrow A \setminus B, A \cup B, A \cap B \in \mathfrak{F}_F$

7] $\mathfrak{F}$ is a σ-Algebra in X that contains all Borel sets in X.

8] $\mathfrak{F}_F = \{E \in \mathfrak{F} : \mu(E) < \infty\}$

9] μ is a measure on $(X, \mathfrak{F})$

10] $\Lambda(f) = \int_X f \, d\mu$ $\forall f \in C_c(X)$

III Regularity properties:

<u>σ-compactness</u> = a metric space (X,d) is σ-compact if $\exists (K_n)_{n=1}^{\infty}$ compact ST $X = \bigcup_{n=1}^{\infty} K_n$

WLOG: assume $K_1 \subset K_2 \subset \dots \subset K_n \subset \dots$ (°°) O/wise $K_2 \to K_2 \cup K_1$, $K_3 \to K_3 \cup K_2 \cup K_1, \dots$

⚠ X compact $\Rightarrow$ X σ-compact — (°°) trivial: $X = \bigcup_{n=1}^{\infty} K_n$ where $K_n = X$ $\forall n$ is compact.

But X locally compact $\not\Rightarrow$ X σ-compact

(°°) Ex: X a set, $d(x,y) = \begin{cases} 1 & x=y \\ 0 & x \neq y \end{cases}$ $\to (X,d)$ locally compact: $\{x\} = B(x,\frac{1}{2})$ $\Rightarrow \{x\} = \mathrm{cl}(B(x,\frac{1}{2}))$ compact

- $K \subset X$ compact $\Leftrightarrow$ K has finitely many pts.
- So X σ-compact $\Rightarrow$ X countable

$\Rightarrow$ X uncountable $\Rightarrow$ X locally compact but NOT σ-compact.

Thm: Let (X,d) be Locally compact + σ-compact metric space:
Let $(\mathfrak{F},\mu)$ be as in the RRT (ie. induced by a positive linear functional Λ on $C_c(X)$).

Then: (a) $\forall E \in \mathfrak{F}, \forall \varepsilon > 0$, $\exists V$ open, $\exists F$ closed ST: $F \subset E \subset V$ and $\mu(V \setminus F) < \varepsilon$

(b) $\forall E \in \mathfrak{F}$: $\mu(E) = \inf\{\mu(V) : V \supset E, V \text{ open}\} = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$ $\Rightarrow$ μ is a Regular measure

$F_\sigma = \bigcup_{n=1}^{\infty}$ closed, $G_\delta = \bigcap_{n=1}^{\infty}$ open

(c) $\forall E \in \mathfrak{F}$: $\exists A$ (F_σ set) and $\exists B$ (G_δ set) ST: $A \subset E \subset B$ and $\mu(B \setminus A) = 0$

$\hookrightarrow$ Apart from sets of measure $= 0$, RRT sets are either F_σ or G_δ

<u>Proof</u>: (a) * if $\mu < \infty$: proved in Step 5
* if $\mu = \infty$: we need to find F, V with $\mu(F) = \infty = \mu(V)$

- X σ-compact $\Rightarrow$ Let $(K_n)_{n=1}^{\infty}$ be an $\uparrow$ seq of compact sets in X, ST $X = \bigcup_{n=1}^{\infty} K_n$; Let $\varepsilon > 0$:

So $(E \cap K_n) \in \mathfrak{F}_F$ and $\exists V_n$ open ST $V_n \supset (E \cap K_n)$ with $\mu(V_n \setminus (E \cap K_n)) < \frac{\varepsilon}{2^{n+1}}$ (step 5) $\to$ Let $V = \bigcup_{n=1}^{\infty} V_n$

So $(V \setminus E) \subset \bigcup_{n=1}^{\infty} V_n \setminus (E \cap K_n)$ (°°) $(V \setminus E) = V \cap E^c = (\bigcup_{n=1}^{\infty} V_n) \cap E^c = \bigcup_{n=1}^{\infty} (V_n \cap E^c) \subset \bigcup_{n=1}^{\infty} (V_n \cap (E \cap K_n)^c)$ since $E \cap K_n \subset E \Rightarrow (E \cap K_n)^c \supset E^c$

$\Rightarrow \boxed{\mu(V \setminus E) < \frac{\varepsilon}{2}}$ (°°) $\mu(V \setminus E) \leq \mu(\bigcup_{n=1}^{\infty} V_n \setminus (E \cap K_n)) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2}$

- We can do the same for E^c: $\exists W$ open ST $W \supset E^c$ and $\mu(W \setminus E^c) < \frac{\varepsilon}{2}$

So set $F = W^c$: then $F \subset E$ and $(E \setminus F) = (W \setminus E^c)$ $\Longrightarrow$ $\boxed{\mu(E \setminus F) < \frac{\varepsilon}{2}}$ (°°) $E \setminus F = E \cap F^c = E \cap W = W \cap (E^c)^c = W \setminus E^c$
So $\mu(E \setminus F) = \mu(W \setminus E^c) < \frac{\varepsilon}{2}$.

- <u>So</u>: F closed, V open, $F \subset E \subset V$ and $\mu(V \setminus E), \mu(E \setminus F) < \varepsilon/2$.

$V \setminus F = (V \setminus E) \cup (E \setminus F)$ (°°) $(V \setminus E) \cup (E \setminus F) = (V \cap E^c) \cup (E \cap F^c) \underset{\substack{E^c \subset F^c \\ E \subset V}}{=} (V \cap E^c \cap F^c) \cup (V \cap E \cap F^c) = (V \cap F^c) \cap (E \cup E^c) = V \cap F^c \cup X = (V \setminus F)$

$\Rightarrow \mu(V \setminus F) \leq \mu(V \setminus E) + \mu(E \setminus F) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

(b) • Part (c) of RRT $\Rightarrow \mu(E) = \inf\{\mu(V) = V \supset E, V \text{ open}\}$ $\forall E \in \mathfrak{F}$ ✓
• if $\mu(E) < \infty$: part (d) of RRT $\Rightarrow \mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$ ✓

<u>Goal</u>: if $\mu(E) = \infty$: $\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$

(°°) Let $\varepsilon = 1$: then (a) $\Rightarrow$ $\exists F \subset E$, F closed ST $\mu(E \setminus F) < 1$; So $\mu(E) = \mu(F) + \mu(E \setminus F) < 1 + \mu(F)$ $\Rightarrow \mu(F) = \infty$

X σ-compact $\Rightarrow$ $\exists K_1 \subset K_2 \subset \dots \subset K_n \subset \dots$ compact ST $X = \bigcup_{n=1}^{\infty} K_n$: So $(F \cap K_n)$ is a $\uparrow$ seq of compact sets $\Rightarrow \mu(F) = \mu(\lim_{n\to\infty}(F \cap K_n))$ $\underset{[\text{monotonicity property}]}{=} \lim_{n\to\infty} \mu(F \cap K_n) = \infty$

So $\mu(F) = \infty = \mu(E)$, hence $\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$

(c) . Part (a) $\Rightarrow$ $\forall n > 0$, $\exists V_n$ open, $\exists F_n$ closed ST $F_n \subset E \subset V_n$ and $\mu(V_n \setminus F_n) < \frac{1}{n}$

• set $\begin{cases} A = \bigcup_{n=1}^{\infty} F_n \to F_\sigma \\ B = \bigcap_{n=1}^{\infty} V_n \to G_\delta \end{cases}$ and $A \subset E \subset B$ ✓

. Also, $(B \setminus A) \subset (V_n \setminus F_n)$ $\forall n$ (°°) $(B \setminus A) = B \cap A^c = (\bigcap_{n=1}^{\infty} V_n) \cap (\bigcap_{n=1}^{\infty} F_n^c) = \bigcap_{n=1}^{\infty} (V_n \cap F_n^c) \subset V_n \cap F_n^c = V_n \setminus F_n$

$\Rightarrow 0 \leq \mu(B \setminus A) \leq \mu(V_n \setminus F_n) < \frac{1}{n}$ $\forall n \in \mathbb{N}$ $\Rightarrow \mu(B \setminus A) = 0$

IV Regularity properties of measurable functions:

Lusin's thm = (X,d) a LCMS, $(\mathcal{F},\mu)$ as in RRT:

Let $f: X \longrightarrow \mathbb{C}$ a MF°, $A \in \mathcal{F}$, ST: $\begin{cases} \mu(A) < \infty \\ f(x) = 0 \ \forall x \notin A \end{cases}$

then: $\forall \varepsilon > 0$, $\exists g \in C_c(X)$ ST $\mu(\{x : f(x) \neq g(x)\}) < \varepsilon$

Note = we can choose $g \in C_c(X)$ ST: $\sup_{x\in X} |g(x)| \leq \sup_{x \in X} |f(x)|$

$(\because)$ $R := \sup_{x\in X} |f(x)| \longrightarrow$ if $R = \infty$ then ✓

Let $R < \infty$: define $\phi(z) = \begin{cases} z & ; \text{if } |z| \leq R \\ R\frac{z}{|z|} & ; \text{if } |z| > R \end{cases}$ ↳ cont!

Let $g_1 \in C_c(X)$ ST $\mu(\{x : f \neq g_1\}) < \varepsilon$

$g := \phi \circ g_1$ is a MF° (ϕ cont, g_1 meas);

Also, $|g(x)| \leq R = \sup_{x\in X} |f(x)|$

$\Rightarrow \mu(\{x : f \neq g\}) = \mu(\{x : \phi(f) \neq \phi(g_1)\}) \leq \underbrace{\mu(\{x : f \neq g_1\})}_{< \varepsilon}$

Note: g_1 cont & $\text{Supp}(g_1) = \text{Supp}(g)$ is compact

$(\because)$ $\{x : g(x) \neq 0\} = \{x : \phi(g_1(x)) \neq 0\} = \{x : g_1(x) \neq 0\}$ as $\phi(z) = 0 \Leftrightarrow z = 0$

Proof:

- **Special case**: A Compact, $0 \leq f(x) < 1$:

Let $0 = s_0 \leq s_1 \leq \ldots \leq s_n \leq \ldots \leq f$ be a seq of simple MF° ST $\lim_{n\to\infty} s_n(x) = f(x)$ $\forall x \in X$

- claim: $\exists T_n$ measurable ST $s_n - s_{n-1} = 2^{-n}\chi_{T_n} \longrightarrow T_n = \bigcup_{k=0}^{2^{n-1}} \{x : \frac{2k+1}{2^n} \leq f(x) \leq \frac{2k+2}{2^n}\}$

Note: $T_n \subset A$ $(\because f(x) > 0$ on $T_n)$

$(\because)$ Let $s_n(x) = \sum_{k=0}^{2^n-1} \frac{k}{2^n} \chi_{\{x : \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}\}}(x) \longrightarrow$ so $s_0 = 0$, $s_1 = \frac{1}{2}\chi_{[0,1)} \Rightarrow s_1 - s_0 = 2^{-1}\chi_{[0,1)}$ ✓

Also: $s_n(x) = \frac{k}{2^n}$ when $\frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} \Rightarrow s_n(x) = \frac{k}{2^n}$ when $\frac{2k}{2^{n+1}} \leq f(x) < \frac{2k+2}{2^{n+1}}$

$k/2^n$ $k+1/2^n$

So $s_{n+1}(x) = \begin{cases} \frac{2k}{2^{n+1}} = \frac{k}{2^n} & , \text{if } \frac{2k}{2^{n+1}} \leq f(x) < \frac{2k+1}{2^{n+1}} \\ \frac{2k+1}{2^{n+1}} & , \text{if } \frac{2k+1}{2^{n+1}} \leq f(x) < \frac{2k+2}{2^{n+1}} \end{cases} \Rightarrow s_{n+1}(x) - s_n(x) = \begin{cases} 0 & , \text{if } \frac{2k}{2^{n+1}} \leq f(x) < \frac{2k+1}{2^{n+1}} \\ \frac{1}{2^{n+1}} & , \text{if } \frac{2k+1}{2^{n+1}} \leq f(x) < \frac{2k+2}{2^{n+1}} \end{cases}$

So $(s_{n+1} - s_n)(x) = \frac{1}{2^{n+1}}\chi_{T_{n+1}}(x)$ For: $T_{n+1} = \bigcup_{k=0}^{2^{n+1}-1} \{x : \frac{2k+1}{2^{n+1}} \leq f(x) \leq \frac{2k+2}{2^n}\}$

- Let $t_n = s_{n+1} - s_n \Rightarrow \boxed{t_n = 2^{-n}\chi_{T_n}}$ $\forall n \geq 1$: so $\forall x \in X$, $f(x) = \sum_{n=1}^{\infty} t_n(x) = \lim_{N\to\infty} \sum_{n=1}^{N} t_n(x) = \lim_{N\to\infty} s_N(x)$ (Telescoping)
- Given $\varepsilon > 0$: $\exists$ compact K_n, $\exists$ open V_n ST $\begin{cases} K_n \subset T_n \subset V_n \subset V \\ \mu(V_n \setminus K_n) < \varepsilon \end{cases}$ where V is fixed and $cl(V)$ is compact

$(\because)$ Let $T_n' = T_n \cap A$:

$f(x) = 0 \ \forall x \notin A \Rightarrow s_n(x) = 0 \ \forall x \notin A \Rightarrow t_n(x) = 0 \ \forall x \notin A \Rightarrow f(x) = \sum_{n=1}^{\infty} t_n(x)$ $\forall x \in T_n'$

$\mu(T_n') < \infty$, A compact $\Rightarrow \exists V$ open ST $A \subset V$, $cl(V)$ compact

$(\because)$ Local compact $\Rightarrow \forall x \in A$, $\exists$ neighb V_x ST $cl(V_x)$ compact $\Rightarrow \{V_x\}_{x\in A}$ open cover of $A \rightarrow$ fin subcov $V_{x_1}, \ldots, V_{x_n}$ $\Rightarrow A \subset V = V_{x_1} \cup \ldots \cup V_{x_n}$ and $cl(V) = \underbrace{cl(V_{x_1}) \cup \ldots \cup cl(V_{x_n})}_{\text{compact}}$

Step 5 $\Rightarrow \exists K_n$ compact, $\exists V_n'$ open ST $K_n \subset T_n' \subset V_n'$ and $\mu(V_n' \setminus K_n) < \varepsilon/2^n$

Let $V_n = V \cap V_n'$: V_n open, $V_n \supset T_n'$ and $\mu(V_n \setminus K_n) \leq \mu(V_n' \setminus K_n) < \varepsilon/2^n \Rightarrow K_n \subset T_n' \subset V_n \subset V$, & $\mu(V_n \setminus K_n) < \varepsilon/2^n$

- Urysohn's Lemma $\Rightarrow \exists h_n$ ST $K_n \prec h_n \prec V_n$
- Let $g(x) = \sum_{n=1}^{\infty} 2^{-n} h_n(x)$: the series converges uniformly $(\because h_n \leq 1)$, and g is cont: $g \in C_c(X)$ and $\text{supp}(g) \subset V$

So $2^{-n}h_n(x) = t_n(x) = 2^{-n}\chi_{T_n}(x)$ except on $V_n \setminus K_n$

So $f(x) = 1 = g(x)$ on K_n $(\because K_n \subset T_n)$ and outside V_n: $f(x) = 0 = g(x)$ $(\because K_n \subset T_n \subset V_n$, $\text{supp}(h_n) \subset V_n)$

$\Rightarrow f(x) = g(x)$ except on $\bigcup_{n=1}^{\infty} V_n \setminus K_n \longrightarrow$ But $\mu(\bigcup_{n=1}^{\infty}(V_n \setminus K_n)) \leq \sum_{n=1}^{\infty} \mu(V_n \setminus K_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$

$\Rightarrow \mu(\{x : f(x) \neq g(x)\}) < \varepsilon$ where $g \in C_c(X)$.

- **Approximation argument**:

- f positive & bounded on A, A compact: $0 < f(x) \leq M$ $\forall x \in A \Rightarrow 0 < \frac{f(x)}{M} \leq 1$ as in the special case above.

So given $\varepsilon > 0$, $\exists g \in C_c(X)$ ST $\mu(\{x : \frac{f(x)}{M} \neq g(x)\}) < \varepsilon \Leftrightarrow \mu(\{x : f(x) \neq M g(x)\}) < \varepsilon$; so $M \cdot g(x) \in C_c(X)$

- f real valued & bounded on A, A compact: $f = f_+ - f_-$ where $\begin{cases} f_+ = \max\{0, f(x)\} \\ f_- = -\min\{0, f(x)\} \end{cases} \rightarrow f_+ \,\&\, f_- = 0$ $\forall x \notin A$, f_+, f_- bounded + positive MF°

So given $\varepsilon > 0$: $\exists g_1, g_2 \in C_c(X)$ ST $\mu(\{x : g_1(x) \neq f_+(x)\}) < \varepsilon/2$, $\mu(\{x : g_2(x) \neq f_-(x)\}) < \varepsilon/2$

So let $g = g_1 - g_2 \in C_c(X)$: $\mu(\{x : g(x) \neq f(x)\}) \leq \mu(\{x : g_1(x) \neq f_+(x)\}) + \mu(\{x : g_2(x) \neq f_-(x)\}) < \varepsilon/2 + \varepsilon/2 = \varepsilon$

- f complex valued & bounded on A, A compact: $f = u + iv$ where u, v real valued & bounded

So given $\varepsilon > 0$: $\exists g_1, g_2 \in C_c(X)$ ST $\mu(\{x : g_1(x) \neq u(x)\}) < \varepsilon/2$, $\mu(\{x : g_2(x) \neq v(x)\}) < \varepsilon/2$

So let $g = g_1 + i g_2 \in C_c(X)$: $\mu(\{x : g(x) \neq f(x)\}) \leq \mu(\{x : g_1(x) \neq u(x)\}) + \mu(\{x : g_2(x) \neq v(x)\}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

- f bounded & measurable on A, $\mu(A) < \infty$:

Let $\varepsilon > 0$: Let K compact ST $\mu(A \setminus K) < \varepsilon/2$: consider $f(x) \cdot \chi_K(x) \rightarrow$ bounded & vanishes outside K, and $(f \cdot \chi_K)(x) = \begin{cases} f(x) & , x \in K \\ 0 & , x \in A^c \end{cases}$

So $\exists g \in C_c(X)$ ST $\mu(\{x : (f\chi_K)(x) \neq g(x)\}) < \varepsilon/2 \Rightarrow \mu(\{x : f(x) \neq g(x)\}) = \mu(\{x \in K \cup A^c : f \neq g\}) + \mu(\{x \in A \setminus K : f \neq g\})$

So $\mu(\{x : f(x) \neq g(x)\}) = \mu(\{x \in K \cup A^c : f\chi_K \neq g\}) + \mu(\{x \in A \setminus K : f \neq g\}) \leq \mu(\{x \in X : f\chi_K \neq g\}) + \mu(A \setminus K) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

- f measurable, $f = 0$ on A^c, $\mu(A) < \infty$: Let $\varepsilon > 0$:

$B_n := \{x : |f(x)| > n\}$: $B_n \subset A$ and $(B_n)_{n=1}^{\infty}$ is a $\downarrow$ seq; so $\bigcap_{n=1}^{\infty} B_n = \phi$ and $\mu(B_1) < \mu(A) < \infty \Longrightarrow \lim_{n\to\infty} \mu(B_n) = 0 \Rightarrow \exists n \in \mathbb{N}$ ST $\mu(B_n) < \frac{\varepsilon}{2}$

$f(x)\chi_{B_n^c}(x)$: bounded & vanishes outside $A \Rightarrow \exists g \in C_c(X)$ ST: $\mu(\{x : f(x)\chi_{B_n^c}(x) \neq g(x)\}) < \varepsilon/2$

So $\mu(\{x \in X : f(x) \neq g(x)\}) = \mu(\{x \in B_n^c : f \neq g\}) + \mu(\{x \in B_n : f \neq g\}) \leq \mu(\{x : f\chi_{B_n^c} \neq g\}) + \mu(B_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Riemann Integral:

I) Definitions:

No regularity assumed!

Setup: $f : [a,b] \longrightarrow \mathbb{R}$ a bounded function : $\begin{cases} m = \inf_{x\in[a,b]} f(x) \\ M = \sup_{x\in[a,b]} f(x) \end{cases} \Rightarrow m \leq f(x) \leq M \quad \forall x \in X$

Partition: a Partition $\mathcal{P}$ of $[a,b]$ is a collection of finitely many pts in $[a,b]$ including a & b.
$\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ where $a = x_0 \leq x_1 \leq \dots \leq x_n = b$

Riemann Sum: Given $\mathcal{P}$, set $\Delta x = x_i - x_{i-1}$, $\begin{cases} m_i = \inf_{x_{i-1}\leq x\leq x_i} f(x) \\ M_i = \sup_{x_{i-1}\leq x\leq x_i} f(x) \end{cases} \Rightarrow \begin{cases} U(\mathcal{P},f) = \sum_{i=1}^{n} M_i \cdot \Delta x_i \longrightarrow \text{upper R-Sum of } f \text{ associated to } \mathcal{P} \\ L(\mathcal{P},f) = \sum_{i=1}^{n} m_i \cdot \Delta x_i \longrightarrow \text{Lower R-Sum of } f \text{ associated to } \mathcal{P} \end{cases}$

Prop: $m\cdot(b-a) \leq L(\mathcal{P},f) \leq U(\mathcal{P},f) \leq M\cdot(b-a)$

Refinment: $\tilde{\mathcal{P}}$ is a refinment of $\mathcal{P}$ if $\mathcal{P} \subset \tilde{\mathcal{P}}$

Prop: $L(\mathcal{P},f) \leq L(\tilde{\mathcal{P}},f) \leq U(\tilde{\mathcal{P}},f) \leq U(\mathcal{P},f)$

Corr: $\mathcal{P}, \mathcal{P}'$ two partitions : $L(\mathcal{P},f) \leq U(\mathcal{P}',f)$ $\quad (\circledast)\ L(\mathcal{P},f) \leq L(\mathcal{P}\cup\mathcal{P}',f) \leq U(\mathcal{P}\cup\mathcal{P}',f) \leq U(\mathcal{P}',f)$

Proof. $\mathcal{P} = \{\overset{a}{x_0}, \dots, \overset{b}{x_n}\}$ and let $\tilde{\mathcal{P}} = \mathcal{P} \cup \{y\}$ with $y \in [a,b]$

Suppose $x_{k-1} \leq y \leq x_k$: let $\begin{cases} M_k' = \sup_{[x_{k-1},y]} f(x) \\ M_k'' = \sup_{[y,x_k]} f(x) \end{cases} \Rightarrow M_k', M_k'' \leq M_k = \sup_{[x_{k-1},x_k]} f(x) \Rightarrow U(\mathcal{P},f) - U(\tilde{\mathcal{P}},f) = M_k \Delta x_k - M_k'\cdot(y - x_{k-1}) - M_k''\cdot(x_k - y) \geq M_k(\Delta x_k - y - \Delta x_k + y) = 0$

So $U(\tilde{\mathcal{P}},f) \leq U(\mathcal{P},f)$ when adding 1 pt $\rightsquigarrow L(\tilde{\mathcal{P}},f) \geq L(\mathcal{P},f)$ (change sign)

Adding finitely many y_n's : $U(\mathcal{P},f) \geq U(\mathcal{P}\cup\{y_1\},f) \geq U(\mathcal{P}\cup\{y_1\}\cup\{y_2\},f) \geq \dots \geq U(\mathcal{P}\cup\{y_1\}\cup\dots\cup\{y_N\},f) = U(\tilde{\mathcal{P}},f)$

Riemann integral = $f : [a,b] \longrightarrow \mathbb{R}$ bounded $\begin{cases} \overline{\int_a^b} f\,dx = \inf_{\mathcal{P}} U(\mathcal{P},f) \longrightarrow \text{Upper R-integral of } f \text{ over } [a,b] \\ \underline{\int_a^b} f\,dx = \sup_{\mathcal{P}} L(\mathcal{P},f) \longrightarrow \text{Lower R-integral of } f \text{ over } [a,b] \end{cases}$

Prop: $m(b-a) \leq \underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx \leq M\cdot(b-a)$ $\quad (\circledast)\ \underline{\int_a^b} f\,dx = \sup_{\mathcal{P}} L(\mathcal{P},f) \leq \inf_{\mathcal{P}'} U(\mathcal{P}',f) = \overline{\int_a^b} f\,dx$

Riemann Integrability: f is Riemann integrable over $[a,b]$ if : $\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$

Riemann Integral: f Riemann integrable : $\int_a^b f\,dx := \underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$

Complex funct°: $f : [a,b] \longrightarrow \mathbb{C}$, $f = u + iv$ R-integrable : $\int_a^b f\,dx := \int_a^b u\,dx + i\int_a^b v\,dx$

II) Properties:

Linearity: $f, g : [a,b] \longrightarrow \mathbb{R}$ R-integrable : $\int_a^b (\alpha f + \beta g)\,dx = \alpha\int_a^b f\,dx + \beta\int_a^b g\,dx \quad \forall \alpha, \beta \in \mathbb{R}$
$f, g : [a,b] \longrightarrow \mathbb{C}$ R-integrable : $\int_a^b (\alpha f + \beta g)\,dx = \alpha\int_a^b f\,dx + \beta\int_a^b g\,dx \quad \forall \alpha, \beta \in \mathbb{C}$

Thm: $f : [a,b] \longrightarrow \mathbb{R}$ a continuous function $\Longrightarrow$ f is R-integrable.

Proof: Let $\varepsilon > 0$: f cont on compact $[a,b] \Rightarrow f$ unif cont on $[a,b] \Rightarrow (\exists \delta > 0)\ (\forall x,y \in [a,b])\ (|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon)$

Let $\mathcal{P}$ be a partit° of $[a,b]$, $\mathcal{P}'$ a ref of $\mathcal{P}$ with $\mathcal{P}' = \{x_0, \dots, x_n\}$ and $\Delta x_i < \delta$; let $\begin{cases} M_i = \sup_{[x_{i-1},x_i]} f(x) = f(t_i) \\ m_i = \inf_{[x_{i-1},x_i]} f(x) = f(s_i) \end{cases}$ (max/min achieved)

So $U(\mathcal{P}',f) - L(\mathcal{P}',f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = \sum_{i=1}^{n} [f(t_i) - f(s_i)]\Delta x_i < \sum_{i=1}^{n} \varepsilon\cdot\Delta x_i = \varepsilon(b-a)$

$\Rightarrow \overline{\int_a^b} f\,dx \leq U(\mathcal{P}',f) < \varepsilon\cdot(b-a) + L(\mathcal{P}',f) \leq \varepsilon\cdot(b-a) + \underline{\int_a^b} f\,dx \Rightarrow 0 \leq \overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx < \varepsilon\cdot(b-a) \longrightarrow 0$ as $\varepsilon \to 0$. ☑

Note: $\mathcal{P}_n$ a seq of partit° : $\mathcal{P}_n = \{x_0, \dots, x_n\}$, $\Delta x_i = \frac{b-a}{2^n}$ $\Rightarrow \mathcal{P}_{n+1} \supset \mathcal{P}_n$ and $U(\mathcal{P}_n,f)\downarrow$, $L(\mathcal{P}_n,f)\uparrow$
$\Rightarrow \lim_{n\to\infty} U(\mathcal{P}_n,f)$ & $\lim_{n\to\infty} L(\mathcal{P}_n,f)$ exist and are equal
↳ bounded seq $\quad$ ↳ unif cont argument above

So $\overline{\int_a^b} f\,dx \leq \lim_{n\to\infty} U(\mathcal{P}_n,f) \leq \lim_{n\to\infty} L(\mathcal{P}_n,f) \leq \underline{\int_a^b} f\,dx \Rightarrow \underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$ ☑

Lebesgue's Problem:

Problem = $f:[a,b]\to\mathbb{R}$ a **bounded** function : when is $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$?

Ex: $f(x) = \chi_{\mathbb{Q}\cap[a,b]}$ is NOT Riemann Integrable $\because \begin{cases} U(\mathcal{P},f)=1 \\ L(\mathcal{P},f)=0 \end{cases} \Rightarrow \begin{cases} \overline{\int_a^b} f(x)dx = 1 \\ \underline{\int_a^b} f(x)dx = 0 \end{cases}$

Lemma = $\exists$ a sequence of partitions of $[a,b]$: $\mathcal{P}_n = \{x_0, x_1, \dots, x_n\}$, $a = x_0 < x_1 < \dots < x_n = b$

ST : • $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ $\forall n$, and $|x_k - x_{k-1}| < \frac{1}{n}$ $\forall x_k, x_{k-1} \in \mathcal{P}_n$.

• $\lim_{n\to\infty} L(\mathcal{P}_n, f) = \underline{\int_a^b} f(x)dx$, and $\lim_{n\to\infty} U(\mathcal{P}_n, f) = \overline{\int_a^b} f(x)dx$

Proof: • $\exists$ seq of partitions of $[a,b]$: $\mathcal{R}_n'$ and $\mathcal{R}_n''$ ST $\lim_{n\to\infty} U(\mathcal{R}_n', f) = \overline{\int_a^b} f(x)dx$ and $\lim_{n\to\infty} \underline{\int_a^b} f(x)dx$

Refine $\mathcal{R}_n'$ and $\mathcal{R}_n''$ ST the distance between successive pts is $< \frac{1}{n}$.

Replace $\begin{cases} \mathcal{R}_n' \text{ by } \mathcal{R}_1' \cup \dots \cup \mathcal{R}_n' \\ \mathcal{R}_n'' \text{ by } \mathcal{R}_1'' \cup \dots \cup \mathcal{R}_n'' \end{cases}$ to get an increasing seq of partitions $\Rightarrow$ set $\mathcal{P}_n = \mathcal{R}_n' \cup \mathcal{R}_n''$

• $\mathcal{P}_n = \{x_0, \dots, x_n\}$ and $M_i = \sup_{x_{i-1}\le x \le x_i} f(x)$ and $m_i = \inf_{x_{i-1}\le x\le x_i} f(x)$ $\Rightarrow \begin{cases} U_n(x) := f(a)\cdot\chi_{\{a\}}(x) + \sum_{i=1}^{k_n} M_i \cdot \chi_{(x_{i-1}, x_i]} \\ L_n(x) := f(a)\cdot\chi_{\{a\}}(x) + \sum_{i=1}^{k_n} m_i \cdot \chi_{(x_{i-1}, x_i]} \end{cases}$

obviously : $L_n(x) \le f(x) \le U_n(x)$ and $L_n(x) \le L_{n+1}(x) \le U_{n+1}(x) \le U_n(x)$ ($\because \mathcal{P}_{n+1} \supset \mathcal{P}_n$)

So we get : $\begin{cases} \int_{[a,b]} U_n\, dm = 0 + \sum_{i=1}^{k_n} M_i \cdot \Delta x_i = U(\mathcal{P}_n, f) \\ \int_{[a,b]} L_n\, dm = 0 + \sum_{i=1}^{k_n} m_i \cdot \Delta x_i = L(\mathcal{P}_n, f) \end{cases}$ and $\begin{cases} \lim_{n\to\infty} L_n(x) := L(x) \\ \lim_{n\to\infty} U_n(x) := U(x) \\ L(x) \le f(x) \le U(x) \quad \forall x\in[a,b] \end{cases}$ exist $\forall x\in[a,b]$ ($\because$) $(U_n)\uparrow$ and $(L_n)\downarrow$ seq and $L(x), U(x)$ are Borel funct° (limits of Borel funct° $\hookrightarrow$ simple MF°)

So: $\begin{cases} \int_{[a,b]} U\, dm \overset{LDCT}{=} \lim_{n\to\infty} \int_{[a,b]} U_n\, dm = \lim_{n\to\infty} U(\mathcal{P}_n, f) = \overline{\int_a^b} f\, dx \\ \int_{[a,b]} L\, dm \overset{LDCT}{=} \lim_{n\to\infty} \int_{[a,b]} L_n\, dm = \lim_{n\to\infty} L(\mathcal{P}_n, f) = \underline{\int_a^b} f\, dx \end{cases}$

$(\exists\varepsilon>0)(\forall\delta>0)(\exists y\in[a,b])(|x-y|<\delta$ and $|f(x)-f(y)|\ge\varepsilon)$

Lebesgue's thm = $f:[a,b]\to\mathbb{R}$ a bounded function, $D = \{x\in[a,b] : f \text{ is discont at } x\}$

• D is a Borel subset of $[a,b]$

• $m(D) = 0 \overset{(1)}{\iff} \overset{(2)}{}$ f is Riemann Integrable on $[a,b]$, and $\int_{[a,b]} f\, dm = \int_a^b f(x)dx$

• if either (1) or (2) holds, then : f is Lebesgue Measurable on $[a,b]$

RK: f may not be Borel Measurable, **But** if (1) or (2) holds :

$\exists$ a Borel Meas. $g(x)$ ST : $g(x) = f(x)$ on a Borel set $B\subset[a,b]$, and $m(B) = b-a$, and $\int_{[a,b]} g\, dm = \int_a^b f\, dx$

Proof: claim : $x\in[a,b]\setminus\bigcup_{n=1}^{\infty}\mathcal{P}_n$ is a continuity pt of $f \iff U(x) = L(x)$

$\boxed{\Rightarrow}$ x Cont pt of f : let $\varepsilon>0 \Rightarrow \exists\delta>0$ ST "$|x-y|<\delta \Rightarrow |f(x)-f(y)|<\frac{\varepsilon}{2}$" $\forall y\in[a,b]$; Let $\frac{1}{n}<\delta$ and $\mathcal{P}_n = \{x_0, \dots, x_{k_n}\}$

$x\notin\mathcal{P}_n \Rightarrow x\in(x_{i-1}, x_i)$: $\forall y\in[x_{i-1}, x_i]$, $|x-y| \le |x_i - x_{i-1}| < \frac{1}{n} < \delta \Rightarrow |f(x)-f(y)| < \varepsilon/2 \Rightarrow f(x)-\frac{\varepsilon}{2} < f(y) < f(x)+\frac{\varepsilon}{2}$

$\begin{cases} L_n(x) = m_i \ge f(x) - \frac{\varepsilon}{2}, & (L_n)\nearrow \\ U_n(x) = M_i \le f(x) + \varepsilon/2, & (U_n)\searrow \end{cases} \Rightarrow \begin{cases} L(x) \ge f(x) + \varepsilon/2 \\ U(x) \le f(x) - \varepsilon/2 \end{cases} \Rightarrow 0 \le U(x) - L(x) \le \varepsilon \quad \forall\varepsilon>0 \Rightarrow U(x) = L(x)$

$\boxed{\Leftarrow}$ $U_n(x) \ge L_n(x)$ and $\lim_{n\to\infty} L_n(x) = L(x) \overset{!}{=} U(x) = \lim_{n\to\infty} U_n(x) \Rightarrow \forall\varepsilon>0 : \exists n\in\mathbb{N}$, ST $0 \le U_n(x) - L_n(x) < \varepsilon$; take $\mathcal{P}_n = \{x_0, \dots, x_{k_n}\}$

Let $x\in(x_{i-1}, x_i)$: $\begin{cases} U_n(x) = \sup_{[x_{i-1},x_i]} f(y) = M_i \\ L_n(x) = \inf_{[x_{i-1},x_i]} f(y) = m_i \end{cases} \Rightarrow \forall y\in[x_{i-1}, x_i] : \begin{cases} f(y)-f(x) \le M_i - m_i < \varepsilon \\ f(x)-f(y) \le M_i - m_i < \varepsilon \end{cases} \Rightarrow |f(y)-f(x)|<\varepsilon$ $\forall y\in[x_{i-1}, x_i]$ and $x\in(x_{i-1}, x_i)$

$\exists\delta>0$ ST : $(x-\delta, x+\delta) \subset (x_{i-1}, x_i)$, So $|y-x|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

• $\tilde{D} := \{x\in[a,b] : U(x) > L(x)\}$ is a Borel set ($\because$) $(U-L)(x) \ge 0$ is a Borel funct° $\Rightarrow \tilde{D} = (U-L)^{-1}(\underset{\to\text{Borel}}{(-\infty,\infty)})$)

claim $\Rightarrow D = (\tilde{D}\setminus\bigcup_{n=1}^{\infty}\mathcal{P}_n) \cup \{x\in\bigcup_{n=1}^{\infty}\mathcal{P}_n : f \text{ discont at } x\}$ is Borel ! ($\because$) countable sets and countable unions of Borel sets are Borel ($\because$) points are Borel sets

• $m(D) = m(\tilde{D})$ ($\because$) $m(\text{countable set}) = 0$ since $m(pt) = 0$

$\boxed{\Rightarrow}$ So $m(D) = 0 \Rightarrow m(\tilde{D}) = 0 = \{x : L(x) \ne U(x)\} \Rightarrow \overline{\int_a^b} f\, dx = \int_{[a,b]} U\, dm = \int_{[a,b]} L\, dm = \underline{\int_a^b} f\, dx \Rightarrow f$ is Riemann Integrable

$\boxed{\Leftarrow}$ f Riemann Integrable $\Rightarrow \int_{[a,b]} L\, dm = \underline{\int_a^b} f\, dx = \overline{\int_a^b} f\, dx = \int_{[a,b]} U\, dm \Rightarrow \int_{[a,b]} (U-L)\, dm = 0$ and $U(x) \ge L(x)$ $\forall x\in[a,b] \Rightarrow m(D) = m(\tilde{D}) = 0$

So far the proof works with m on Borel σ-Alg : now need Lebesgue σ-Alg to show that f is Lebesgue meas

• Suppose (1) or (2) holds : Let $V\subset\mathbb{R}$ open ; $x\in\tilde{D}^c \Rightarrow U(x) = f(x) \Rightarrow m(\tilde{D}) = 0 \Rightarrow$ outside a set of Lebesgue measure 0, f coincides with a Borel funct°

So $f^{-1}(V) = (f^{-1}(V)\cap\tilde{D}^c) \cup (f^{-1}(V)\cap\tilde{D}) = (U^{-1}(V)\cap\tilde{D}^c) \cup (f^{-1}(V)\cap\tilde{D})$ = Lebesgue meas

$\hookrightarrow \{f : U(x) = f(x)\}$ $\hookrightarrow$ Borel measurable $\hookrightarrow$ Lebesgue measurable but may not be Borel meas

($\because$) $m(\tilde{D}) = 0 \Rightarrow m(f^{-1}(V)\cap\tilde{D}) \le m(\tilde{D}) = 0$ by Completeness of measures on Lebesgue σ-Alg

So f is Lebesgue measurable $\Rightarrow \begin{cases} U(x) = f(x), & x\notin\tilde{D} \\ U(x) \ge f(x), & x\in\tilde{D} \end{cases}$ with $m(\tilde{D}) = 0$, $\Rightarrow \int_{[a,b]} f\, dm = \int_{[a,b]} U\, dm = \int_a^b f(x)dx$

Lebesgue Measure on $[a,b]$:

① Definition:

- $X = [a,b]$, $d(x,y) = |x-y|$ $\Rightarrow$ (X,d) is a compact metric space.
- $C_c(X) = C(X)$: Vspace of all cont. functions : $f : [a,b] \longrightarrow \mathbb{C}$
- $\Lambda : C(X) \longrightarrow \mathbb{C}$ a linear functional ST : $\Lambda(f) = \int_a^b f(x)\,dx$ (Riemann Integral)

Apply RRT = ($\because$) X compact $\Rightarrow$ X locally compact ; Λ linear & positive

$\exists$ a σ-Algebra $\mathfrak{F}$ in $X = [a,b]$ containing all Borel Sets

$\exists$ a unique measure m on $(X, \mathfrak{F})$

ST: (a) $\Lambda(f) = \int_a^b f\,dx = \int_{[a,b]} f\,dm \quad \forall f \in C(X)$

(b) $m(E) = \sup\{m(K) : K \subset E, K \text{ compact}\} = \inf\{m(V) : V \supset E, V \text{ open}\}$

(c) Let $E \in \mathfrak{F}$ and $m(E) = 0$: if $A \subset E$, then $m(A) = 0$ and $A \in \mathfrak{F}$

Def: • m is called the Lebesgue Measure on $[a,b]$

• $\mathfrak{F}$ is called the Lebesgue σ-Algebra in $[a,b]$

• $f : [a,b] \rightarrow \mathbb{C}$ is Lebesgue measurable if f is measurable WRT the Lebesgue σ-Algebra.

⚠ on $\mathfrak{F}$: Lebesgue(Lebesgue(x)) is NOT always Lebesgue MF.
on $\mathcal{B}$: Borel(Borel(x)) is Borel MF.

Note: the restriction of m to the Borel σ-Algebra is also called a Lebesgue measure.

⚠ This restriction is uniquely specified on $\mathcal{B}$ by (a) and (b), BUT (c) does NOT hold on $\mathcal{B}$.

Ⓘ Properties:

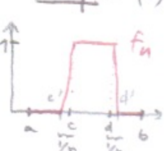
Prop. (1) $m([c,d]) = d-c$, $\forall [c,d] \subset [a,b]$

(2) $m(\{x\}) = 0$, $\forall x \in [a,b]$

(3) $m([c,d]) = m([c,d)) = m((c,d]) = m((c,d)) = d-c$, $\forall a \le c \le d \le b$ ($\because$) use (2)

(4) m is uniquely specified on $\mathfrak{F}$ by (3) and the regularity property : $m(E) = \inf\{m(V) : V \supset E, V \text{ open}\}$, $\forall E \in \mathfrak{F}$

Proof: (1) • m is a finite measure : ($\because$) $\mathbb{1}(x) = x \Rightarrow \Lambda(\mathbb{1}) = \int_a^b dx = b-a \Rightarrow m([a,b]) = \int_{[a,b]} \mathbb{1}\,dm = b-a < \infty$



• If $a < c < d < b$:

$\Lambda(f_n) = \int_a^b f_n\,dx = (d-c) + \frac{2}{n} \Rightarrow \lim_{n\to\infty} \Lambda(f_n) = \lim_{n\to\infty} \int_a^b f_n\,dx = d-c$

$\lim_{n\to\infty} f_n = \chi_{[c,d]} \Rightarrow \int_{[a,b]} (\lim_{n\to\infty} f_n)\,dm = \int_{[a,b]} \chi_{[c,d]}\,dm = \int_{[c,d]} dm = m([c,d])$

$\Rightarrow m([c,d]) = \int_{[a,b]} (\lim_{n\to\infty} f_n)\,dm \overset{\text{LDCT}}{=} \lim_{n\to\infty} \int_a^b f_n\,dx = d-c$

• if $c = a$, $d < b$:

• if $c > a$, $d = b$:

(2) Let $a < x < b$: $\exists N > 0$ ST $A_n = [x - \frac{1}{n}, x + \frac{1}{n}] \subset [a,b]$, $\forall n \ge N$, Hence $\bigcap_{n=N}^{\infty} A_n = \{x\}$ and $(A_n)_{n=N}^{\infty}$ is a ↘ Seq

m is finite ($m(A_N) < \infty$) $\Rightarrow m(\{x\}) = m(\bigcap_{n=N}^{\infty} A_n) = \lim_{n\to\infty} m(A_n) = \lim_{n\to\infty} \frac{2}{n} = 0$

(4). Let $V \subset [a,b]$ open : $\exists\, O \subset \mathbb{R}$ open ST $V = O \cap [a,b]$

So $O = \bigcup_{n=1}^{\infty} (a_n, b_n)$ for disjoint $(a_n, b_n) \subset \mathbb{R}$

$\Big\}\ V = \left(\bigcup_{n=1}^{\infty} (a_n,b_n)\right) \cap [a,b] = \bigcup_{n=1}^{\infty} \underbrace{(a_n,b_n) \cap [a,b]}_{I_n \text{ : an interval}}$

• (a_n, b_n) disjoint $\Rightarrow I_n$ disjoint $\Rightarrow m(V) = \sum_{n=1}^{\infty} m(I_n) = \sum_{n=1}^{\infty} |I_n|$

↳ So if we know m on intervals (a_n, b_n), we know m on any open set $V \subset [a,b]$

$\Rightarrow$ m is uniquely specified on open sets

• $m(E) = \inf\{m(V) : V \supset E, V \text{ open}\}$ specifies m uniquely on $\mathfrak{F}$.

Lebesgue Measure on $\mathbb{R}$:

- $X = \mathbb{R}$, $d(x,y) = |x-y|$ $\Rightarrow (X, d)$ is a Locally Compact + σ-Compact metric Space
- $C_c(\mathbb{R})$: Vspace of all continuous functions $f: \mathbb{R} \to \mathbb{C}$ with Compact Support ($\exists [a,b] \subset \mathbb{R}$, ST $Supp(f) \subset [a,b]$)
- $\Lambda : C_c(\mathbb{R}) \to \mathbb{C}$ a positive linear functional ST: $\Lambda(f) = \int_{-\infty}^{\infty} f(x)dx = \int_a^b f(x)dx$ if $Supp(f) \subset [a,b]$

Apply RRT = (∵) X locally compact, σ-compact metric Space ; Λ linear & positive

$\exists$ a σ-Algebra $\mathfrak{F}$ in $\mathbb{R}$ containing all Borel sets
$\exists$ a unique measure m on $(\mathbb{R}, \mathfrak{F})$

ST: (a) $\Lambda(f) = \int_{-\infty}^{\infty} f(x)dx \overset{!}{=} \int_{\mathbb{R}} f\, dm$, $\forall f \in C_c(\mathbb{R})$

(b) $m(K) < \infty$, $\forall K \subset \mathbb{R}$ compact

(c) $m(E) = \sup\{m(K) : K \subset E, K \text{ Compact}\} = \inf\{m(V) : V \supset E, V \text{ open}\}$, $\forall E \in \mathfrak{F}$

(d) Let $E \in \mathfrak{F}$ and $m(E) = 0$: if $A \subset E$, then $m(A) = 0$ and $A \in \mathfrak{F}$

Note: From RRT, (c) holds when $m(E) < \infty$
If $m(E) = \infty$, (c) will still hold!

(∵) Let $m(E) = \infty$: $\exists F \subset E$ closed ST $m(F) = \infty$
Set $K_n = F \cap [-n, n]$ $\Rightarrow K_n$ is Compact and $\lim_{n\to\infty} m(K_n) = m(F) = \infty$

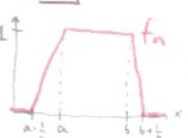
Prop = (1) $m([a,b]) = b-a$, $\forall [a,b] \subset \mathbb{R}$

(2) $m(\{x\}) = 0$, $\forall x \in \mathbb{R}$

(3) $m([a,b]) = m([a,b)) = m((a,b]) = m((a,b)) = b-a$, $\forall a \leq b$ in $\mathbb{R}$ (∵) use (2)

(4) m is uniquely Specified on $\mathfrak{F}$ by (3), and the regularity property : $m(E) = \inf\{m(V) : V \supset E, V \text{ open}\}$, $\forall E \in \mathfrak{F}$

Proof: (1) Same argument as for intervals :



$\Lambda(f_n) = (b-a) + \frac{1}{n}$ $\Rightarrow \lim_{n\to\infty} \Lambda(f_n) = b-a = \int_{\mathbb{R}} \chi_{[a,b]} dm = m([b-a])$

(2) Same as previous

(3) Same as previous

(4) $m((a,\infty)) = \lim_{n\to\infty} m(a, a+n) = \infty$
$m((-\infty, a)) = \lim_{n\to\infty} m(a-n, a) = \infty$
$\Rightarrow m((a,b)) = b-a$ even if $a = \infty$ or $b = \infty$

if $V \subset \mathbb{R}$ is open : $V = \bigcup_{n=1}^{\infty} (a_n, b_n)$ disjoint $\Rightarrow m(V) = \sum_{n=1}^{\infty} (b_n - a_n)$ $\Rightarrow m$ is uniquely specified $\forall V \subset \mathbb{R}$ open

Regularity property $\Rightarrow m$ is uniquely Specified $\forall E \in \mathfrak{F}$

Translation = fix $a \in \mathbb{R}$: $\Phi_a : \mathbb{R} \to \mathbb{R}$, $x \mapsto x+a$, and $\begin{cases} \Phi_a \circ \Phi_b = \Phi_{a+b}, & \forall a,b \in \mathbb{R} \\ \Phi_a^{-1} = \Phi_{-a}, & \forall a \in \mathbb{R} \end{cases}$

Translation Invariance = $E \in \mathfrak{F} \Leftrightarrow \Phi_a(E) \in \mathfrak{F}$ (Lebesgue measurable), and in that case : $m(E) = m(\Phi_a(E))$

Prop: Translation invariance uniquely characterizes Lebesgue Measures on $\mathbb{R}$:

Let $\tilde{m}$ be a measure on $(\mathbb{R}, \mathcal{B})$ ST : $\begin{cases} \bullet\ \tilde{m}(\Phi_a(E)) = \tilde{m}(E), \forall a \in \mathbb{R} \text{ and } \forall E \in \mathcal{B} \\ \bullet\ \tilde{m}(E) = \inf\{\tilde{m}(V) : V \supset E, V \text{ open}\}, \forall E \in \mathcal{B} \end{cases}$

then: $\tilde{m}(E) = c \cdot m(E)$, $\forall E \in \mathcal{B}$, where $c = \tilde{m}([0,1))$

Proofs: (See Xin's note + annexe)

Continuity = Let $E \in \mathfrak{F}$ and α ST: $0 < \alpha < m(E)$
then: $\exists K \subset E$ Compact ST $m(K) = \alpha$

Proof : $m(E) = \sup\{m(K') : K' \subset E, K' \text{ compact}\} \Rightarrow \exists K' \subset E$ Compact ST $m(K') > \alpha$

- Define : $f(x) := m(K' \cap [-x, x])$: $|f(x) - f(y)| \leq 2 \cdot |x-y| \Rightarrow f$ is continuous

(∵) WLOG : $x > y$, $\Delta x = x - y > 0$; write : $[-x, x] = [-y, y] \cup [-y-\Delta x, -y) \cup (y, y+\Delta x]$

so: $m(K' \cap [-x,x]) = m(K' \cap [-y,y]) + m(K' \cap [-y-\Delta x, -y)) + m(K' \cap (y, y+\Delta x])$ (∵) disjoint

$\Rightarrow f(x) - f(y) = m(\cdot\cdot) + m(\cdot\cdot) \leq \Delta x + \Delta x = 2\Delta x$

- $f(0) = 0$ and $\lim_{x\to\infty} f(x) = m(K') > \alpha > 0$ $\Rightarrow$ Intermediate Value thm : $\exists x_\alpha > 0$ ST $f(x_\alpha) = \alpha = m(\underbrace{K' \cap [-x_\alpha, x_\alpha]}_{:= K \text{ Compact!}})$

Restrictions of Measures:

Generalities: $(X, \mathcal{F}, \mu)$ a measure space: $Y \subset \mathcal{F}$

- $\mathcal{F}_Y = \{E \cap Y : E \in \mathcal{F}\}$ is a σ-Algebra in Y

  (as) $(E \cap Y) \in \mathcal{F}_Y \Rightarrow (E \cap Y)^c = (E^c \cap Y) \in \mathcal{F}_Y$

  $(E_n \cap Y) \in \mathcal{F}_Y \Rightarrow \bigcup_{n=1}^{\infty} (E_n \cap Y) = (\bigcup_{n=1}^{\infty} E_n) \cap Y \in \mathcal{F}_Y$

  $\hookrightarrow$ so $F \in \mathcal{F}_Y \Leftrightarrow F = E \cap Y$ for some $E \in \mathcal{F}$
- $\mu_Y(F) = \mu(E \cap Y)$ defines a measure on $(Y, \mathcal{F}_Y)$

$\Rightarrow$ $(Y, \mathcal{F}_Y, \mu_Y)$ is the restriction of $(X, \mathcal{F}, \mu)$ to Y

Ex: Lebesgue measure on $\mathbb{R}$ $\longrightarrow$ Lebesgue measure on $[a,b]$

Riemann-Stieltjes Integral:

Setup: $[a,b] \subset \mathbb{R}$, $f : [a,b] \longrightarrow \mathbb{R}$ a bounded function, $\alpha : [a,b] \longrightarrow \mathbb{R}$ an increasing function

$\alpha(x) \leq \alpha(y) \quad \forall x \leq y$

Partition: $\mathcal{P} = \{x_0, \dots, x_n\}$ a partition of $[a,b]$, $\begin{cases} \Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \\ \sum_{i=1}^{n} \Delta\alpha_i = \alpha(b) - \alpha(a) \end{cases}$ and $\begin{cases} m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x) \\ M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x) \end{cases}$

RS Sums: $L(\mathcal{P}, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$, $U(\mathcal{P}, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$

Prop:
- $m \cdot (\alpha(b) - \alpha(a)) \leq L(\mathcal{P}, f, \alpha) \leq U(\mathcal{P}, f, \alpha) \leq M \cdot (\alpha(b) - \alpha(a))$, where $m \leq f(x) \leq M \quad \forall x \in [a,b]$
- $\mathcal{P} \subset \mathcal{P}' \Rightarrow L(\mathcal{P}, f, \alpha) \leq L(\mathcal{P}', f, \alpha) \leq U(\mathcal{P}', \alpha, f) \leq U(\mathcal{P}, \alpha, f)$

RS integral: $\underline{\int_a^b} f\,d\alpha := \sup_{\mathcal{P}} L(\mathcal{P}, f, \alpha)$; $\overline{\int_a^b} f\,d\alpha := \inf_{\mathcal{P}} U(\mathcal{P}, f, \alpha)$;

Prop: $m \cdot (\alpha(b) - \alpha(a)) \leq \underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha \leq M \cdot (\alpha(b) - \alpha(a))$

RS Integrability: Given α: $RS(\alpha) = \{f : \underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha\}$ $\longrightarrow$ collection of all RS integrable functions for a given α.

Prop: $f : [a,b] \to \mathbb{R}$: f continuous $\Longrightarrow f \in RS(\alpha)$, $\underline{\underline{\forall \alpha}} : [a,b] \to \mathbb{R}$ increasing

Proof:
- if $\alpha(b) = \alpha(a)$: $\Delta\alpha_i = 0 \Rightarrow \underline{\int_a^b} f\,d\alpha = 0 = \overline{\int_a^b} f\,d\alpha$ ✓
- if $\alpha(b) > \alpha(a)$: $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in [a,b])(|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ (as) f cont $\Rightarrow f$ unif cont on $[a,b]$.

  Let $\mathcal{P}$ be a partition st $\Delta x_i < \delta \Rightarrow U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \frac{\varepsilon}{\alpha(b) - \alpha(a)} \sum_{i=1}^{n} \Delta x_i = \varepsilon \Rightarrow U(\mathcal{P}, f, \alpha) < \varepsilon + L(\mathcal{P}, f, \alpha)$

  So $\overline{\int_a^b} f\,d\alpha = \inf U(\mathcal{P}, f, \alpha) < \varepsilon + L(\mathcal{P}, f, \alpha) \leq \varepsilon + \underline{\int_a^b} f\,d\alpha \Rightarrow 0 \leq \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \varepsilon \quad \underline{\forall \varepsilon > 0}$!

Prop:
- $c \in \mathbb{R} : f \in RS(\alpha) \Longrightarrow (c \cdot f) \in RS(\alpha)$ and $\int_a^b (cf)\,d\alpha = c \cdot \int_a^b f\,d\alpha$
- $f, g \in RS(\alpha) \Longrightarrow (f+g) \in RS(\alpha)$ and $\int_a^b (f+g)\,d\alpha = \int_a^b f\,d\alpha + \int_a^b g\,d\alpha$

Proof:
- Exercise
- Let $\mathcal{P}, \mathcal{P}'$ be 2 partitions of $[a,b]$: $L(\mathcal{P}, f, \alpha) + L(\mathcal{P}', g, \alpha) \leq L(\mathcal{P} \cup \mathcal{P}', f, \alpha) + L(\mathcal{P} \cup \mathcal{P}', g, \alpha) \leq L(\mathcal{P} \cup \mathcal{P}', f+g, \alpha) \leq \underline{\int_a^b} (f+g)\,d\alpha \leq \overline{\int_a^b} (f+g)\,d\alpha$

  $\inf(f) + \inf(g) \leq \inf(f+g)$

  So $L(\mathcal{P}, f, \alpha) + L(\mathcal{P}', g, \alpha) \leq \underline{\int_a^b} (f+g)\,d\alpha \leq \overline{\int_a^b} (f+g)\,d\alpha \leq U(\mathcal{P}, f, \alpha) + U(\mathcal{P}', g, \alpha)$

$\begin{cases} \text{Take sup over } \mathcal{P}, \text{ then sup over } \mathcal{P}' \text{ for } \underline{\text{LHS}}: & \int_a^b f\,d\alpha + \int_a^b g\,d\alpha \leq \cdots \\ \text{Take inf over } \mathcal{P}, \text{ then inf over } \mathcal{P}' \text{ for } \underline{\text{RHS}}: & \cdots \leq \int_a^b f\,d\alpha + \int_a^b f\,d\alpha \end{cases}$

$\Rightarrow \int_a^b f\,d\alpha + \int_a^b g\,d\alpha \leq \underline{\int_a^b} (f+g)\,d\alpha \leq \overline{\int_a^b} (f+g)\,d\alpha \leq \int_a^b f\,d\alpha + \int_a^b g\,d\alpha \Rightarrow \begin{cases} \int_a^b (f+g)\,d\alpha = \int_a^b f\,d\alpha + \int_a^b g\,d\alpha \\ (f+g) \in RS(\alpha). \end{cases}$

Complex valued f: $f : [a,b] \longrightarrow \mathbb{C}$: $f = u + iv$ with $u, v : [a,b] \longrightarrow \mathbb{R}$ continuous

then $\int_a^b f\,d\alpha := \int_a^b u\,d\alpha + i \int_a^b v\,d\alpha$

Prop: $f, g : [a,b] \longrightarrow \mathbb{C}$ continuous $\Longrightarrow \int_a^b (c_1 f + c_2 g)\,d\alpha = c_1 \int_a^b f\,d\alpha + c_2 \int_a^b g\,d\alpha$, $\forall c_1, c_2 \in \mathbb{C}$

Lebesgue-Stieltjes measure on $[a,b]$:

(I) Definition & Basic Properties:

- $X = [a,b]$, $d(x,y) = |x-y| \implies (X,d)$ is a compact metric space $\implies X$ locally compact
- $C_c([a,b]) = C([a,b])$: Vspace of all cont. functions $f : [a,b] \longrightarrow \mathbb{C}$
- $\Lambda : C([a,b]) \longrightarrow \mathbb{C}$ a linear functional ST : $\Lambda(f) = \int_a^b f\,d\alpha$ $\longrightarrow$ RS integral

Apply RRT: $\exists$ a σ-Algebra $\mathfrak{F}$ in $[a,b]$ containing all Borel sets
$\exists$ a unique measure m_α on $([a,b], \mathfrak{F})$ ST :

ST: (a) $\Lambda(f) = \int_a^b f\,d\alpha \doteq \int_{[a,b]} f\,dm_\alpha \quad \forall f \in C([a,b])$

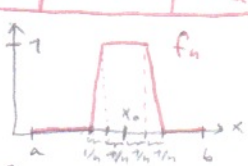
(b) $m_\alpha(E) = \sup\{m_\alpha(K) : K \subset E, K \text{ compact}\} = \inf\{m_\alpha(V) : V \supset E, V \text{ open}\}$, $\forall E \in \mathfrak{F}$

(c) Let $E \in \mathfrak{F}$, $m(E) = 0$: if $A \subset E$, then $m(A) = 0$ and $A \in \mathfrak{F}$ $\longrightarrow$ m_α is a complete measure

Def: $\alpha : [a,b] \longrightarrow \mathbb{R}$ is increasing : $\alpha(x-) := \lim_{y \uparrow x} \alpha(y)$; $\alpha(x+) = \lim_{y \downarrow x} \alpha(y)$

Prop: α Cont at $x_0 \iff \alpha(x_0+) = \alpha(x_0-)$

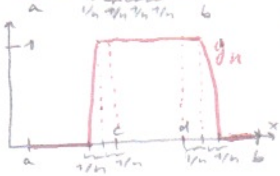
Properties of m_α: (1) $m_\alpha(\{x_0\}) = \alpha(x_0+) - \alpha(x_0-)$, $\forall x_0 \in (a,b)$
$m_\alpha(\{a\}) = \alpha(a+) - \alpha(a)$, and $m_\alpha(\{b\}) = \alpha(b) - \alpha(b-)$



(2) $\alpha : [a,b] \longrightarrow \mathbb{R}$: α increasing $\implies$ α has at most countably many discontinuities.

(3) $m_\alpha([c,d]) = \alpha(d+) - \alpha(c-)$, $\forall a < c < d < b$

(4) $m_\alpha([c,d)) = \alpha(d-) - \alpha(c-)$; $m_\alpha((c,d]) = \alpha(d+) - \alpha(c+)$; $m_\alpha((c,d)) = \alpha(d-) - \alpha(c+)$



Proof: (1) $\bullet$ $\lim_{n\to\infty} f_n(x) = \chi_{\{x_0\}} \implies \lim_{n\to\infty} \int_{[a,b]} f_n\,dm_\alpha \overset{\text{LDCT}}{=} m_\alpha(\{x_0\})$

$\bullet$ $\left.\begin{array}{l} \int_{[a,b]} f_n\,dm_\alpha = \int_a^b f_n\,d\alpha \geq \alpha(x_0 + \frac{1}{n}) - \alpha(x_0 - 1/n) \\ \int_{[a,b]} f_n\,dm_\alpha = \int_a^b f_n\,d\alpha \leq \alpha(x_0 + \frac{2}{n}) - \alpha(x_0 - 2/n) \end{array}\right\} \implies m_\alpha(\{x_0\}) = \lim_{n\to\infty} \int_{[a,b]} f_n\,dm = \alpha(x_0+) - \alpha(x_0-)$

(2) $D_k := \{x \in [a,b] : \alpha(x+) - \alpha(x-) \geq 1/k\}$, $D = \bigcup_{k=1}^\infty D_k$ = set of discont of f $\longrightarrow$ Goal : D_k is countable.

$\left.\begin{array}{l} \bullet \text{ Let } a < x_1 < \dots < x_n < b \text{ be distinct pts in } D_k \\ \bullet \text{ Let } \varepsilon > 0 \text{ ST } x_\ell - \varepsilon > x_{\ell-1} + \varepsilon \ \forall \ell = 1, \dots, n \end{array}\right\} \implies \alpha(b) \geq \alpha(x_n + \varepsilon) \underbrace{- \alpha(x_n - \varepsilon) + \alpha(x_{n-1} + \varepsilon)}_{<0} + \dots + \alpha(x_1+) \underbrace{- \alpha(x_1-) + \alpha(a)}_{<0} \geq \frac{n}{k} + \alpha(a)$

$\varepsilon \to 0$: $\alpha(b) \geq \alpha(x_n+) - \alpha(x_n-) + \dots + \alpha(x_1+) - \alpha(x_1-) + \alpha(a) \geq \frac{n}{k} + \alpha(a)$

$\implies \alpha(b) - \alpha(a) \geq \frac{n}{k} \implies n \leq k \cdot (\alpha(b) - \alpha(a)) \longrightarrow D_k$ is countable!

(3) $\lim_{n\to\infty} g_n(x) = \chi_{[c,d]} \implies m_\alpha([c,d]) = \alpha(d+) - \alpha(c-)$ (similar argument than (1)).

(4) $m_\alpha((c,d]) = m_\alpha([c,d]) - m_\alpha(\{c\}) = [\alpha(d+) - \alpha(c-)] - [\alpha(c+) - \alpha(c-)] = \alpha(d+) - \alpha(c+)$

Thm: $f : [a,b] \longrightarrow \mathbb{R}$ bounded, D = set of discont of f, $\alpha : [a,b] \longrightarrow \mathbb{R}$ increasing :
$m_\alpha(D) = 0 \implies f \in RS(\alpha)$, and $\int_a^b f\,d\alpha = \int_{[a,b]} f\,dm_\alpha$ $\longrightarrow$ f is LS measurable

Ex: $x_0 \in (a,b)$, $f(x) = \begin{cases} 1, & x = x_0 \\ 0, & x \neq x_0 \end{cases}$, $\alpha(x) = \begin{cases} 1, & a \leq x \leq x_0 \\ 0, & x_0 < x \leq b \end{cases}$ $\implies L(P,f,\alpha) = 0 \neq 1 = U(P,f,\alpha) \to f \notin RS(\alpha)$ BUT: $\int_{[a,b]} f\,dm_\alpha = m_\alpha(\{x_0\}) = \alpha(x_0+) - \alpha(x_0-)$

Converse : $\alpha : [a,b] \longrightarrow \mathbb{R}$ increasing and continuous
$f \in RS(\alpha) \implies m_\alpha(D) = 0$ and $\int_a^b f\,d\alpha = \int_{[a,b]} f\,dm_\alpha$

Generalization: $\bullet$ $\alpha : [a,b] \longrightarrow \mathbb{R}$ of finite variation, f has finite nb of discont on $[a,b]$: $\int_a^b f\,d\alpha$ is defined

HW7

$\bullet$ $\alpha : [a,b] \longrightarrow \mathbb{R}$ of bounded variation, f piecewise continuous :
Then $\alpha = \alpha_1 - \alpha_2$ where $\alpha_1, \alpha_2 \uparrow$ (Jordan decomp : $\frac{V_a^x(\alpha) \pm \alpha(x)}{2}$)
and $\int_a^b f\,d\alpha := \int_{[a,b]} f\,dm_{\alpha_1} - \int_{[a,b]} f\,dm_{\alpha_2}$

Lebesgue - Stieltjes measure on $\mathbb{R}$:

(I) Definition & Basic Properties :

- $X = \mathbb{R}$, $d(x,y) = |x-y|$ $\rightarrow$ $(\mathbb{R}, d)$ is a locally compact metric space
- $C_c(\mathbb{R})$: Vspace of all cont functions $f : \mathbb{R} \rightarrow \mathbb{C}$ with compact support
- $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ an increasing function (at most countably many discont)
- $\Lambda : C_c(\mathbb{R}) \rightarrow \mathbb{C}$ a positive linear functional ST : $\Lambda(f) = \int_{-\infty}^{\infty} f \, d\alpha = \int_a^b f \, d\alpha$, $\forall f \in C_c(\mathbb{R})$, when $\text{supp}(f) \subset [a,b]$

Apply RRT : $\exists$ a σ-Algebra $\mathcal{F}$ in $\mathbb{R}$ containing all Borel sets

$\exists$ a unique measure m_α on $(\mathbb{R}, \mathcal{F})$

ST : (a) $\Lambda(f) = \int_{-\infty}^{\infty} f \, d\alpha \doteq \int_{\mathbb{R}} f \, dm_\alpha$, $\forall f \in C_c(\mathbb{R})$

(b) $m_\alpha(K) < \infty$, $\forall K$ compact

(c) $m_\alpha(E) = \sup\{m(K) : K \subset E, K \text{ compact}\} = \inf\{m(V) : V \supset E, V \text{ open}\}$, $\forall E \in \mathcal{F}$

(d) Let $E \in \mathcal{F}$, $m(E) = 0$: if $A \subset E$, then $m(A) = 0$ and $A \in \mathcal{F}$ $\rightarrow$ m_α is a complete measure

Properties of m_α : (1) $m_\alpha(\{x_0\}) = \alpha(x_0+) - \alpha(x_0-)$, $\forall x_0 \in \mathbb{R}$

(2) $\forall [a,b] \subset \mathbb{R}$ finite : $m_\alpha([a,b]) = \alpha(b+) - \alpha(a-)$, $m_\alpha([a,b)) = \alpha(b-) - \alpha(a-)$

$m_\alpha((a,b)) = \alpha(b-) - \alpha(a+)$, $m_\alpha((a,b]) = \alpha(b+) - \alpha(a+)$

(II) Application To probability :

Setup : $(X, \mathcal{F}, \mu)$ a probability space : $\mu(X) = 1$

Random Variable : $f : X \longrightarrow \mathbb{R}$

Distribution measure of f : f a r.v. : $\mu_f(E) := \mu(f^{-1}(E))$, $\forall E \subset \mathbb{R}$

Prop : μ_f defines a probability measure

Ex : . $f \sim \mathcal{N}(0,1)$ r.v : $\mu_f(E) = \frac{1}{\sqrt{2\pi}} \int_E e^{-\frac{x^2}{2}} dm(x)$, $\forall E \subset \mathbb{R}$

. $f \sim \text{Poisson}(\lambda)$ r.v : $m_f(\{n\}) = \frac{\lambda^n}{n!} e^{-\lambda}$ and $\mu_f(E) = \sum_{n \in E} m_f(\{n\})$, $\forall E \subset \mathbb{R}$

Distribution function of f : $F(x) := \mu_f((-\infty, x])$

Prop : . F is positive and right continuous

. $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$

. $F(b) - F(a) = \mu_f((a,b])$, $\forall a < b \in \mathbb{R}$

Basic result : $\mu_f = \mu_F$

Proof : . Both are equal on $(a,b]$ $\rightarrow$ $\mu_f(V) = \mu_F(V)$ $\forall V \subset \mathbb{R}$ open

. Extend it to $E \in \mathcal{F}$: $\mu_f(E) = \inf\{\mu_f(V) : V \supset E, V \text{ open}\}$

$\mu_F(E) = \inf\{\mu_F(V) : V \supset E, V \text{ open}\}$

Lebesgue Measure in $\mathbb{R}^d$:

I. σ-Compactness & regularity:

Lemma: X a metric space in which every open set is σ-compact.
Let: λ be a positive Borel measure ST $\lambda(K) < \infty$ $\forall K \subset X$ compact.
Then: λ is regular

II. Preliminaries: $\mathbb{R}^d$ is σ-compact (∵) Take balls centered at the rationals

Box: A set $W \subset \mathbb{R}^d$ ST: $W := [\alpha_1, \beta_1) \times \cdots \times [\alpha_d, \beta_d)$
Volume: $\text{Vol}(W) := \prod_{i=1}^{d} (\beta_i - \alpha_i)$

δ-Box: Let $\delta > 0$, $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$: $Q_\delta(a) := [a_1, a_1+\delta) \times \cdots \times [a_d, a_d+\delta)$
Prop: $\text{Vol}(Q_\delta(a)) = \delta^d$

Def:
- $P_n := \{x \in \mathbb{R}^d : x_i = \alpha_i 2^{-n} \text{ and } \alpha_i \in \mathbb{Z}\}$ ⟶ set of all pts whose coords are integer multiples of 2^{-n}
- $\Omega_n := \{Q_{2^{-n}}(x) : x \in P_n\}$ ⟶ Collect° of all boxes $Q_{2^{-n}}(x)$ with $x \in P_n$

Prop:
(a) Ω_n is countable (∵) $\cong \mathbb{Z}^d$
(b) $\mathbb{R}^d = \bigcup_{Q \in \Omega_n} Q$ and: $Q \cap Q' = \phi$ $\forall Q \neq Q'$ in Ω_n
(c) $Q \in \Omega_n$: $\text{Vol}(Q) = 2^{-nd}$, and $r > n \Rightarrow |P_r \cap Q| = 2^{(r-n)d}$
(d) $r < n$: $Q \in \Omega_n$, $Q' \in \Omega_r \Rightarrow Q \subset Q'$ or $Q \cap Q' = \phi$ (∵) Same endpts
(e) $\forall V \subset \mathbb{R}^d$ open & nonempty: V is a countable disjoint union of boxes in $\bigcup_{i=1}^{\infty} \Omega_i$

III. Existence of a Lebesgue measure:

Thm: $\exists$ a measure m defined on a σ-Algebra $\mathcal{M}$ on $\mathbb{R}^d$ ST:
(a) $m(Q) = \text{Vol}(Q)$ $\forall$ boxes Q
(b) $\mathcal{M} \supset \mathcal{B}$ and m is regular on $\mathcal{M}$
(c) m is translation invariant: $\forall E \in \mathcal{M}$, $\forall x \in X$: $m(\overbrace{E+x}^{\{x+y : y \in E\}}) = m(E)$
(d) If μ is a positive Borel translation invariant measure on $\mathbb{R}^d$:
then $\exists c > 0$ ST $\mu(E) = c \cdot m(E)$ $\forall E$ Borel.
(e) $\forall$ linear transformation $T: \mathbb{R}^d \to \mathbb{R}^d$: $\exists \Delta(T) \in \mathbb{R}$ ST $m(T(E)) = \Delta(T) \cdot m(E)$ $\forall E \in \mathcal{M}$
In particular: T is a rotation $\Longrightarrow \Delta(T) = 1$

Note:
- m = "Lebesgue measure on $\mathbb{R}^d$"
- $\mathcal{M}$ = "Lebesgue σ-Algebra on $\mathbb{R}^d$"

Notes on determinant.

For Any linear transfo $T:\mathbb{R}^d \longrightarrow \mathbb{R}^d$, $T(e_i) = \sum_{j=1}^{d} \alpha_{ij} e_j$

so the matrix representation of T: $[T] = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1d} \\ \vdots & \ddots & \vdots \\ \alpha_{d1} & \cdots & \alpha_{dd} \end{bmatrix}$

If $T = T_1 \circ T_2$, then $\det(T) = \det(T_1) \cdot \det(T_2)$

<u>claim:</u> $\Delta(T_1 T_2) = \Delta(T_1) \cdot \Delta(T_2)$

(pf) $m((T_1 T_2)(E)) = m(T_1(T_2(E))) = \Delta(T_1) \cdot m(T_2(E))$
$= \Delta(T_1) \cdot \Delta(T_2) \cdot m(E)$.

(1) $T(\{e_1, \dots, e_d\}) = \{e_1, \dots, e_d\}$ (could be a permutation)
$|\det(T)| = 1$. Let $W = \bigtimes_{i=1}^{d} [a_i, b_i)$
obvious that: $m(T(W)) = m(W) = 1 \cdot m(W) \Rightarrow \Delta(T) = 1$.
(Swap vertices of box)

(2) $T(e_i) = \alpha e_i$, and: $T(e_j) = e_j \ \forall j \neq i \Rightarrow \det(T) = \alpha$.
$m(T(W)) = |\alpha| \cdot m(W) \Rightarrow \Delta(T) = |\alpha|$.

(3) $T(e_1) = e_1 + e_2$, $T(e_i) = e_i \ \forall i = 2, \dots, d \Rightarrow \det(T) = 1$.
So $m(T(W)) = 1 \cdot m(W) \rightarrow \Delta(T) = 1$.

Part 3:

Complex & Signed Measures

- Complex measures
- Signed measures
- Lebesgue - Radon - Nikodym theorem & applications
- Fundamental theorem of Calculus

Complex measures:

$(X, \mathcal{F})$ a measurable space:

Complex measure: A function $\mu : \mathcal{F} \longrightarrow \mathbb{C}$, ST: $\forall$ family of disjoint measurable sets $\{E_n\}_{n=1}^{\infty}$: $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$ with $\sum_{n=1}^{\infty} |\mu(E_n)| < \infty$.

Prop: • $|\mu(E)| < \infty, \ \forall E \in \mathcal{F}$

- If μ is a positive measure: μ is a complex measure $\Leftrightarrow \mu(X) < \infty$

• $\mu(\phi) = 0$

Signed measure: A complex measure that takes real values only: $\mu : \mathcal{F} \longrightarrow \mathbb{R}$

Prop: $\{z_k\}_{k=1}^{\infty} \in \mathbb{C}$, $\phi : \mathbb{N} \to \mathbb{N}$ a bijection

then $\sum_{k=1}^{\infty} z_k = \sum_{k=1}^{\infty} z_{\phi(k)} \ \underline{\forall \phi} \Leftrightarrow \sum_{k=1}^{\infty} |z_k| < \infty$

$\hookrightarrow$ permutation

Partition: Given $E \in \mathcal{F}$: a family of M-sets $\{E_n\}_{n=1}^{\infty} \in \mathcal{F}$ is a partition of E if: $\begin{cases} E = \bigcup_{n=1}^{\infty} E_n \\ E_n \cap E_m = \phi \ \ \forall n \neq m \end{cases}$

Total variation of μ: $|\mu| : \mathcal{F} \longrightarrow [0, \infty]$, $|\mu|(E) = \sup\limits_{\substack{\text{All partitions} \\ \{E_n\} \text{ of } E}} \left(\sum_{n=1}^{\infty} |\mu(E_n)| \right)$

Prop: • $\mu(\phi) = 0$ (pf) $E_n = \phi \Rightarrow \mu(\phi) = \sum_{n=1}^{\infty} \mu(\phi) = \lim_{N \to \infty} (N \cdot \mu(\phi)) \Rightarrow \mu(\phi) = 0$.

- $|\mu(\phi)| = 0$

Collection of all complex measures: $\mathcal{M}(X) = \{\mu : \mu \text{ is a complex measure on } (X, \mathcal{F})\}$

Prop: • $\mathcal{M}(X)$ is a Vector space over $\mathbb{C}$

(pf) • $\mu_1, \mu_2 \in \mathcal{M}(X)$: $(\mu_1 + \mu_2)(E) = \mu_1(E) + \mu_2(E)$ is also a complex meas.
• $\mu \in \mathcal{M}(X), c \in \mathbb{C}$: $(c\mu)(E) = c \cdot \mu(E)$ is a complex measure.

- $\|\mu\| := |\mu|(X)$ defines a norm on $\mathcal{M}(X)$
- $(\mathcal{M}(X), \|\cdot\|)$ is a Banach Space

Thm: $|\mu|$ is a measure on $(X, \mathcal{F})$ $\longrightarrow \forall \{E_n\}_{n=1}^{\infty} \in \mathcal{F}$ disjoint with $E = \bigcup_{n=1}^{\infty} E_n$: $|\mu|(E) = \sum_{n=1}^{\infty} \mu(E_n)$

Proof: • claim: $|\mu|(E) \geq \sum_{n=1}^{\infty} |\mu|(E_n)$

(pf) • let $t_n \geq 0$ ST: $|\mu|(E_n) \geq t_n \geq 0$. then $\exists$ partit° $\{E_{n,m}\}_{m=1}^{\infty}$ of E_n ST: $|\mu|(E_n) \geq \sum_{m=1}^{\infty} |\mu(E_{n,m})| \geq t_n$

- Family $\{E_{n,m}\}_{n,m=1}^{\infty}$ is a partition of E: $|\mu|(E) \geq \sum_{n,m=1}^{\infty} |\mu(E_{n,m})| = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\mu(E_{n,m})| \geq \sum_{n=1}^{\infty} t_n$

So: $|\mu|(E) \geq \sum_{n=1}^{N} t_n \ \ \underline{\underline{\forall N \in \mathbb{N}}}$: take $t_n = |\mu|(E_n) \Rightarrow |\mu|(E) \geq \sum_{n=1}^{N} |\mu|(E_n) \xRightarrow[N \to \infty]{} : |\mu|(E) \geq \sum_{n=1}^{\infty} |\mu|(E_n)$

• claim: $|\mu|(E) \leq \sum_{n=1}^{\infty} |\mu|(E_n)$

(pf) • Let $\{A_j\}_{j=1}^{\infty}$ a partit° of E: $\{A_j \cap E_n\}_{j=1}^{\infty}$ is a partit° of E_n ; $\{A_j \cap E_n\}_{n=1}^{\infty}$ is a partition of A_j.

- So: $\sum_{j=1}^{\infty} |\mu(A_j)| = \sum_{j=1}^{\infty} \left|\mu\left(\bigcup_{n=1}^{\infty} A_j \cap E_n\right)\right| \underset{\uparrow}{=} \sum_{j=1}^{\infty} \left|\sum_{n=1}^{\infty} \mu(A_j \cap E_n)\right| \leq \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} |\mu(A_j \cap E_n)| \leq \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |\mu(A_j \cap E_n)| \leq \sum_{n=1}^{\infty} |\mu|(E_n)$

μ countably additive

- So: $\sum_{j=1}^{\infty} |\mu(A_j)| \leq \sum_{n=1}^{\infty} |\mu|(E_n)$ $\underline{\underline{\forall}}$ partition $\{A_j\}_{j=1}^{\infty}$ of E $\xRightarrow[\text{part° } \{A_j\}]{\text{Sup over all}}$ $|\mu|(E) \leq \sum_{n=1}^{\infty} |\mu|(E_n)$.

↳ Nice trick!

Lemma: $\forall z_1, \dots, z_n \in \mathbb{C}$: $\exists S \subset \{1, \dots, n\}$, ST $\left|\sum_{k\in S} z_k\right| \geq \frac{1}{\pi}\sum_{k=1}^{n}|z_k|$

Proof: • write $z_k = |z_k| e^{i\theta_k}$, $-\pi \leq \theta_k \leq \pi$: Let $S(\theta) = \{k : \cos(\theta_k - \theta) \geq 0\}$

• claim: $\left|\sum_{k\in S(\theta)} z_k\right| \geq \sum_{k=1}^{N}|z_k| \cos^+(\theta_k - \theta)$, where $\cos^+(\theta_k-\theta) = \max\{0, \cos(\theta_k-\theta)\} = \frac{|\cos(\theta_k-\theta)| + \cos(\theta_k-\theta)}{2}$

(*) $Re(e^{-i\theta} z_k) = Re(e^{-i\theta} e^{i\theta_k}|z_k|) = Re(|z_k| e^{i(\theta_k-\theta)}) = |z_k| \cdot |\cos(\theta_k - \theta)|$

So $\left|\sum_{k\in S(\theta)} z_k\right| \underset{|e^{i\theta}|=1}{=} \left|\sum_{k\in S(\theta)} e^{-i\theta} z_k\right| \geq \sum_{k\in S(\theta)} Re(e^{-i\theta} z_k) = \sum_{k\in S(\theta)} |z_k| \cdot \underbrace{|\cos(\theta_k-\theta)|}_{\geq 0 \text{ on } S(\theta)} = \sum_{k\in S(\theta)} |z_k| \cdot \cos(\theta_k - \theta)$

$= \sum_{k=1}^{n} |z_k| \cdot \cos^+(\theta_k - \theta)$

• the funct° $\theta \overset{\in[-\pi,\pi]}{\longmapsto} \sum_{k=1}^{n} |z_k| \cdot \cos^+(\theta_k - \theta)$ is continuous & achieves its max/min at some $\theta_0 \in [-\pi, \pi]$

$\Rightarrow \sum_{k=1}^{n} |z_k| \cos^+(\theta_k - \theta) \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \left[\sum_{k=1}^{n}|z_k| \cos^+(\theta_k-\theta)\right] = \frac{1}{2\pi}\sum_{k=1}^{n}|z_k| \int_{-\pi}^{\pi} \frac{|\cos(\theta_k-\theta)| + \cos(\theta_k-\theta)}{2} d\theta$

$\Rightarrow \left|\sum_{k\in S(\theta)} z_k\right| \geq \frac{1}{\pi}\sum_{k=1}^{n}|z_k|$ ☑ $\qquad = \frac{1}{2\pi}\sum_{k=1}^{n}|z_k| \int_{-\pi}^{\pi} \frac{|\cos(\theta_k-\theta)|}{2} d\theta = \frac{1}{\pi}\sum_{k=1}^{n}|z_k|$

Thm: μ a complex measure on $(X, \mathcal{F}) \Rightarrow |\mu|(X) < \infty$

Proof: Assume $|\mu|(X) = \infty$ and let $t = \pi \cdot (1 + |\mu(X)|) < \infty$ (*) $|\mu(X)| < \infty$

Construct°: $|\mu|(X) = \infty > t \Rightarrow \exists$ a partition $\{E_n\}_{n=1}^{\infty}$ of X, ST: $\sum_{n=1}^{\infty}|\mu(E_n)| > t$, and $\exists N > 0$ ST: $\sum_{n=1}^{N}|\mu(E_n)| > t$

Let $z_n = \mu(E_n)$ and $S \subset \{1, \dots, N\}$ as the prev lemma

$\Rightarrow \left|\sum_{n\in S} \mu(E_n)\right| \geq \frac{1}{\pi}\sum_{n=1}^{N}|\mu(E_n)| > \frac{t}{\pi}$.

Step 1: • $A_0 := \bigcup_{n\in S} E_n$: $|\mu(A_0)| > \frac{t}{\pi} = 1 + |\mu(X)| > 1$

• $B_0 := X \setminus A_0$: $\mu(X) = \mu(A_0) + \mu(B_0) \Rightarrow |\mu(B_0)| = |\mu(X) - \mu(A_0)| \geq |\mu(A_0)| - |\mu(X)| > \underbrace{\frac{t}{\pi} - |\mu(X)|}_{=1}$

⇨ If $|\mu|(X) = \infty$, $\exists A_0 B_0 \in \mathcal{F}$ disjoint ST $X = A_0 \cup B_0$, $\begin{cases}|\mu(A_0)| > 1 \\ |\mu(B_0)| > 1\end{cases}$

Also $|\mu|(A_0) + |\mu|(B_0) \underset{\hookrightarrow |\mu| \text{ is a measure}}{=} |\mu|(X) = \infty \Rightarrow$ Assume WLOG that $|\mu|(B_0) = \infty$.

Step 2: Partition $B_0 = A_1 \cup B_1$, $A_1 \cap B_1 = \phi$, $\begin{cases}|\mu(A_1)| > 1 \\ |\mu(B_1)| > 1\end{cases}$ and $|\mu|(B_1) = \infty$

Step 3: Partition $B_1 = A_2 \cup B_2 \dots$

⋮

Result: get a sequence of sets $\{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty}$ ST A_n's are disjoint and $|\mu(A_n)| > 1$ $\forall n \in \mathbb{N}$

Set $E = \bigcup_{n=1}^{\infty} A_n$: so $\mu(E) = \sum_{n=1}^{\infty}\mu(A_n)$ and we must have $\sum_{n=1}^{\infty}|\mu(A_n)| < \infty$ as μ is a complex measure

But $|\mu(A_n)| > 1$ $\forall n \Rightarrow \lim_{n\to\infty}|\mu(A_n)| \geq 1 \neq 0 \Rightarrow \sum_{n=1}^{\infty}|\mu(A_n)| = \infty$ $\boxed{\Rightarrow\Leftarrow}$

Signed Measures:

① Jordan decomposition:

<u>Total variation of μ</u> = $|\mu|(E) = \sup\limits_{\text{all partit}^{\circ}} \left(\sum\limits_{n=1}^{\infty} |\mu(E_n)| \right)$

<u>Positive variat° of μ</u> = $\mu^+ = \frac{1}{2}(|\mu| + \mu)$

<u>Negative variat° of μ</u> = $\mu^- = \frac{1}{2}(|\mu| - \mu)$

$\Big\}$ positive finite measures (∵) $|\mu|(E) \geq |\mu(E)| \quad \forall E \in \mathcal{F}$

<u>Prop</u>: • $\mu = \mu^+ - \mu^-$
• $|\mu| = \mu^+ + \mu^-$
$\Big\}$ Jordan decomposit° of μ

<u>Prop</u>: • $\{E_n\}_{n=1}^{\infty}$ an <u>increasing</u> sequence of measurable sets: $E = \bigcup\limits_{n=1}^{\infty} E_n \Rightarrow \mu(E) = \lim\limits_{n\to\infty} \mu(E_n)$
• $\{E_n\}_{n=1}^{\infty}$ a <u>decreasing</u> sequence of measurable sets: $E = \bigcap\limits_{n=1}^{\infty} E_n \Rightarrow \mu(E) = \lim\limits_{n\to\infty} \mu(E_n)$

<u>Proof</u>: $\mu = \mu^+ - \mu^-$: μ^+ & μ^- are positive <u>finite</u> measures $\Rightarrow \begin{cases} \mu^+(E) = \lim\limits_{n\to\infty} \mu(E_n) < \infty \\ \mu^-(E) = \lim\limits_{n\to\infty} \mu(E_n) < \infty \end{cases}$ When $E_n \nearrow$ or $E_n \searrow$
$\hookrightarrow \mu^+(E_1) < \infty, \ \mu^-(E_1) < \infty$

$\Rightarrow \mu(E) = \mu^+(E) - \mu^-(E) = \lim\limits_{n\to\infty} \mu^+(E_n) - \lim\limits_{n\to\infty} \mu^-(E_n) = \lim\limits_{n\to\infty} (\mu^+(E_n) - \mu^-(E_n)) = \lim\limits_{n\to\infty} \mu(E_n)$

<u>Thm</u>: $\exists E_*, E^* \in \mathcal{F}$ ST : (1) $\mu(E^*) = \sup\limits_{E\in\mathcal{F}} \mu(E)$
(2) $\mu(E_*) = \inf\limits_{E\in\mathcal{F}} \mu(E)$

<u>Proof</u>: Sufficient to prove (1) : (2) follows : $\exists E_* \in \mathcal{F}$ ST $-\mu(E_*) = \sup\limits_{E\in\mathcal{F}} (-\mu(E)) = -\inf\limits_{E\in\mathcal{F}} \mu(E) \Rightarrow \mu(E_*) = \inf\limits_{E\in\mathcal{F}} \mu(E)$ ☑

(1) Let $L = \sup\limits_{E\in\mathcal{F}} \mu(E)$: then $0 \leq L < \infty$ (∵) $\mu(E) \leq |\mu(E)| \leq |\mu|(E) \leq |\mu|(X) < \infty$

• By def of the supremum : $\exists (E_n)_{n=1}^{\infty} \in \mathcal{F}$ ST : $L = \lim\limits_{n\to\infty} \mu(E_n)$; Set $E = \bigcup\limits_{n=1}^{\infty} E_n$

• <u>claim</u>: $E = \bigcup\limits_{m=1}^{2^n} E_{n,m}$ ST $E_{n,m} \in \mathcal{F}$ are disjoint, and $E_{n,m} = \bigcap\limits_{j=1}^{n} S_j$ where $S_j = E_j$ or $E \setminus E_j$ (See notes for proof)

• Let $B_n = \bigcup\limits_{m:\mu(E_{n,m})\geq 0} E_{n,m}$: So $\mu(B_n) \geq 0$, and $\mu(B_n) \geq \mu(E_n) \ \forall n \in \mathbb{N}$

(∵) $E_n = E \cap E_n = \left(\bigcup\limits_{m=1}^{2^n} E_{n,m}\right) \cap E_n = \bigcup\limits_{m=1}^{2^n} (E_{n,m} \cap E_n) = \bigcup\limits_{\text{some } m\text{'s}} E_{n,m}$ because $E_{n,m} = \bigcap\limits_{j=1}^{n} S_j$ ($\to E_j$ or $E\setminus E_j$) $\Rightarrow E_{n,m} \cap E_n = \begin{cases} E_{n,m}, & \text{if } S_n = E_n \\ \emptyset, & \text{if } S_n = E \setminus E_n \end{cases}$

So $\mu(E_n) = \sum\limits_{\text{some } m\text{'s}} \mu(E_{n,m}) \leq \sum\limits_{m:\mu(E_{n,m})\geq 0} \mu(E_{n,m}) = \mu(B_n)$

• Set $C_{n,k} = B_n \cup \dots \cup B_{n+k}$: $(C_{n,k})_{k=1}^{\infty}$ is an $\nearrow$ seq. <u>Claim</u>: $\mu(C_{n,k}) \leq \mu(C_{n,k+1}) \ \forall k$

(∵) $C_{n,k+1} \setminus C_{n,k} = (B_n \cup \dots \cup B_{n+k+1}) \setminus (B_1 \cup \dots \cup B_n) \subset B_{n+k+1} = \bigcup\limits_{m:\mu(E_{n+k+1,m})\geq 0} E_{n+k+1,m} \rightarrow$ So $\mu(E_{n+k+1,m}) \geq 0$

Pick $E_{n+k+1,m}$ ST : $E_{n+k+1,m} \cap (C_{n,k+1} \setminus C_{n,k}) \neq 0$ (if such a set $\nexists$, $\mu(C_{n,k+1}) = \mu(C_{n,k})$ Trivially)

So $E_{n+k+1,m} = \bigcap\limits_{j=1}^{n+k+1} S_j$ where $S_j = E_j$ or $E \setminus E_j \Rightarrow E_{n+k+1,m} \subset \bigcap\limits_{j=1}^{n+p} S_j \quad \forall 0 \leq p \leq k$

<u>Note</u>: $\bigcap\limits_{j=1}^{n+p} S_j$ is not a partit° of B_{n+p} ((∵) o'wise $E_{n+k+1,m} \subset B_{n+p} \subset C_{n,k}$ $\boxed{\Rightarrow\Leftarrow}$ as $E_{n+k+1,m} \cap (C_{n,k+1} \setminus C_{n,k}) \neq \emptyset$)

$E_{n+k+1,m} \cap (C_{n,k+1} \setminus C_{n,k}) \neq 0 \Rightarrow E_{n+k+1,m} \subset B_{n+k+1}$ and $E_{n+k+1,m} \subset C_{n,k+1} \setminus C_{n,k}$

So $C_{n,k+1} \setminus C_{n,k} = \bigcup\limits_{E_{n+k+1,m} \cap (C_{n,k+1}\setminus C_{n,k}) \neq \emptyset} E_{n+k+1,m} \Rightarrow \mu(C_{n,k+1} \setminus C_{n,k}) \underset{\text{positive measure}}{=} \sum\limits_{\substack{E_{n+k+1,m} \cap (C_{n,k+1}\setminus C_{n,k}) \neq \emptyset \\ \mu(E_{n+k+1,m}) \geq 0}} \mu(E_{n+k+1,m}) \geq 0$ and $C_{n,k+1} = C_{n,k} \cup (C_{n,k+1} \setminus C_{n,k})$ disjoint

So $\mu(C_{n,k+1}) = \mu(C_{n,k}) + \mu(C_{n,k+1} \setminus C_{n,k}) \geq \mu(C_{n,k})$

• Set $A_n = \bigcup\limits_{k=0}^{\infty} C_{n,k}$: $(C_{n,k})_{k=1}^{\infty}$ is $\nearrow \Rightarrow \mu(A_n) = \lim\limits_{k\to\infty} \mu(C_{k,n})$; and $\left.\begin{matrix} \mu(C_{n,k}) \geq \mu(B_n) \\ \mu(C_{n,k+1}) \geq \mu(C_{n,k}) \end{matrix}\right\} \Rightarrow \mu(A_n) \geq \mu(B_n) \geq \mu(E_n)$

• Let $E^* = \bigcap\limits_{n=1}^{\infty} A_n$: $A_n = \bigcup\limits_{k=0}^{\infty} B_{n+k} = \bigcup\limits_{j=n}^{\infty} B_j$

But $C_{n,0} = B_n$, $C_{n,k} = B_n \cup \dots \cup B_{n+k} \Rightarrow \bigcup\limits_{k=0}^{\infty} C_{n,k} = B_n \cup B_{n+1} \cup \dots$

So $(A_n)_{n=1}^{\infty}$ is a $\searrow$ seq $\Rightarrow \mu(E^*) = \lim\limits_{n\to\infty} \mu(A_n) \geq \lim\limits_{n\to\infty} \mu(E_n) = L = \sup\limits_{E\in\mathcal{F}} \mu(E) \Rightarrow \mu(E^*) = \sup\limits_{E\in\mathcal{F}} \mu(E)$ ☑

(IV) Positive & Negative variation of μ:

Def: • $\tilde{\mu}_+(E) = \sup\{\mu(A) : A \subset E, A \in \mathcal{F}\}$ $\Rightarrow 0 \le \tilde{\mu}_+(E) < \infty$

• $\tilde{\mu}_-(E) = -\inf\{\mu(A) : A \subset E, A \in \mathcal{F}\}$ $\Rightarrow 0 \le \tilde{\mu}_-(E) < \infty$

Thm: $\exists D \subset \mathcal{F}$, ST: $\begin{cases} \tilde{\mu}_+(E) = \mu(E \cap D^c) \\ \tilde{\mu}_-(E) = -\mu(E \cap D) \end{cases}$

Corr: $\tilde{\mu}_+$ and $\tilde{\mu}_-$ are positive measures, and: $\mu = \tilde{\mu}_+ - \tilde{\mu}_-$

(pf) $\tilde{\mu}_+(\bigcup_{n=1}^{\infty} E_n) = \mu((\bigcup_{n=1}^{\infty} E_n) \cap D^c) = \sum_{n=1}^{\infty} \mu(E_n \cap D^c) = \sum \tilde{\mu}_+(E_n)$ ($\{E_n\}$ disjoint)

$\Rightarrow \tilde{\mu}_+(E) - \tilde{\mu}_-(E) = \mu(E \cap D^c) + \mu(E \cap D) = \mu(E)$

Proof: Let $D = E_*$ where $\mu(E_*) = \inf_{E \in \mathcal{F}} \mu(E)$:

• claim: $\forall E \in \mathcal{F}$, $\mu(E \cap D) \le 0$ and $\mu(E \cap D^c) \ge 0$

(pf) $D = (D \cap E) \cup (D \setminus (D \cap E))$ disjoint $\Rightarrow \mu(D) = \mu(D \cap E) + \mu(D \setminus (D \cap E))$

So $\begin{cases} \mu(D \cap E) = \mu(D) - \mu(D \setminus (D \cap E)) \le 0 \text{ by def of infimum.} \\ \mu(D) \le \mu(D \cup E) = \mu((D \cup E) \cap (D \cup D^c)) = \mu(D \cup (E \cap D^c)) = \mu(D) + \mu(E \cap D^c) \Rightarrow \mu(E \cap D^c) \ge 0. \end{cases}$

• Fix $E \in \mathcal{F}$, let $A \subset E$:

$\mu(A) = \mu((A \cap D) \cup (A \cap D^c)) = \overbrace{\mu(A \cap D)}^{\le 0} + \mu(A \cap D^c) \le \mu(A \cap D^c) \le \mu(A \cap D^c) + \overbrace{\mu((E \setminus A) \cap D^c)}^{\ge 0} = \mu((A \cap D^c) \cup ((E \setminus A) \cap D^c))$ $[A \subset E]$ $= \mu(E \cap D^c)$

$\Rightarrow \mu(A) \le \mu(E \cap D^c)$ $\forall A \subset E$ $\Rightarrow \sup\{\mu(A) : A \subset E, E \in \mathcal{F}\} \le \mu(E \cap D^c) \Rightarrow \tilde{\mu}_+(E) \le \mu(E \cap D^c)$

Also, $(E \cap D^c) \subset E \Rightarrow \tilde{\mu}_+(E) = \sup\{\mu(A) : A \subset E, E \in \mathcal{F}\} \ge \mu(E \cap D^c)$

$\underline{\tilde{\mu}_+(E) = \mu(E \cap D^c)}$

• Fix $E \in \mathcal{F}$, let $A \subset E$:

$\mu(A) = \mu((A \cap D) \cup (A \cap D^c)) = \mu(A \cap D) + \overbrace{\mu(A \cap D^c)}^{\ge 0} \ge \mu(A \cap D) \ge \mu(A \cap D) + \overbrace{\mu((E \setminus A) \cap D)}^{\le 0} = \mu(E \cap D)$

$\Rightarrow \mu(A) \ge \mu(E \cap D)$ $\forall A \subset E$ $\Rightarrow -\tilde{\mu}_-(E) = \inf\{\mu(A) : A \subset E, A \in \mathcal{F}\} \ge \mu(E \cap D)$

Also: $(E \cap D) \subset E \Rightarrow -\tilde{\mu}_-(E) = \inf\{\mu(A) : A \subset E, A \in \mathcal{F}\} \le \mu(E \cap D)$

$\underline{\tilde{\mu}_-(E) = -\mu(E \cap D).}$

Thm: $|\mu| = \tilde{\mu}_+ + \tilde{\mu}_-$, and $\tilde{\mu}_+ = \mu_+$, $\tilde{\mu}_- = \mu_-$

Proof: • claim: $|\mu| = \tilde{\mu}_+ + \tilde{\mu}_-$

(pf) Let $D = E_*$: for a partition $E = (E \cap D) \cup (E \cap D^c)$: $|\mu|(E) = \sup_{\{E_n\}} \sum |\mu(E_n)| \ge |\mu(E \cap D)| + |\mu(E \cap D^c)|$

So $|\mu|(E) \ge -\mu(E \cap D) + \mu(E \cap D^c) = \tilde{\mu}_-(E) + \tilde{\mu}_+(E)$

Let $\{E_n\}$ a partit° of E: $\sum_{n=1}^{\infty} |\mu(E_n)| = \sum_{n=1}^{\infty} |\mu((E_n \cap D) \cup E_n \cap D^c))| \le \sum_{n=1}^{\infty} |\mu(E_n \cap D)| + |\mu(E_n \cap D^c)|$
$= \sum_{n=1}^{\infty} \tilde{\mu}_+(E_n) + \tilde{\mu}_-(E_n) = \tilde{\mu}_+(E) + \tilde{\mu}_-(E)$

So $|\mu|(E) = \sup_{\{E_n\}} \sum_{n=1}^{\infty} |\mu(E_n)| \le \tilde{\mu}_+(E) + \tilde{\mu}_-(E)$

$\Rightarrow$ $|\mu|(E) = \tilde{\mu}_+(E) + \tilde{\mu}_-(E)$

• $\begin{cases} |\mu| = \tilde{\mu}_+ + \tilde{\mu}_- \\ |\mu| = \mu_+ + \mu_- \end{cases} \Rightarrow \tilde{\mu}_+ + \tilde{\mu}_- = \mu_+ + \mu_-$

• $\begin{cases} \mu = \tilde{\mu}_+ - \tilde{\mu}_- \\ \mu = \mu_+ - \mu_- \end{cases} \Rightarrow \tilde{\mu}_+ - \tilde{\mu}_- = \mu_+ - \mu_-$

$\Rightarrow \begin{cases} \tilde{\mu}_+ = \mu_+ \\ \tilde{\mu}_- = \mu_- \end{cases}$

Minimality of the Jordan decomposit°: Let $\mu = \nu^+ - \nu^-$ be a decomposit° of μ into two positive measures.

Then: $\forall E \in \mathcal{F}$, $\begin{cases} \nu^+(E) \ge \mu^+(E) \\ \nu^-(E) \ge \mu^-(E) \end{cases}$

Proof: $\mu = \nu^+ - \nu^- = \mu^+ - \mu^- \Rightarrow \nu^+ - \mu^+ = \nu^- - \mu^- \Rightarrow$ Goal: $\nu^+ - \mu^+ \ge 0$, i.e. $\nu^+ \ge \mu^+$ ($\nu^- \ge \mu^-$ follows)

Notice: $\forall E \in \mathcal{F}$, $\nu^+(E) = \mu(E) + \underbrace{\nu^-(E)}_{\ge 0} \ge \mu(E)$

Fix E: $\nu^+(E) \underset{\substack{\nu^+ \text{ is a} \\ \text{positive measure}}}{=} \sup_{\substack{A \in \mathcal{F} \\ A \subset E}} \nu^+(A) \ge \sup_{\substack{A \in \mathcal{F} \\ A \subset E}} \mu(A) = \mu^+(E)$ $\Rightarrow \nu^+(E) \ge \mu^+(E)$ $\forall E \in \mathcal{F}$ $\boxed{\checkmark}$

Lebesgue - Radon - Nikodym Theorem :

(I) Preliminaries :

Let ϕ and μ be measures on $(X, \mathcal{F})$:

Absolute continuity: ϕ is absolutely continuous WRT μ $(\phi \ll \mu)$ if : $\mu(E)=0 \Rightarrow \phi(E)=0 \quad \forall E \in \mathcal{F}$.

Concentration: μ is concentrated on a set $A \in \mathcal{F}$ if : $\mu(E) = \mu(E \cap A) \ \forall E \in \mathcal{F}$

Prop: μ concentrated on $A \in \mathcal{F} \iff \mu(A^c) = 0$

Mutual Singularity: μ and ϕ are mutually singular $(\phi \perp \mu)$ if: μ & ϕ are concentrated on disjoint sets.

Prop:
(1) $\phi_1 \ll \mu$ and $\phi_2 \ll \mu \Longrightarrow (\phi_1 + \phi_2) \ll \mu$
(2) $\phi_1 \perp \mu$ and $\phi_2 \perp \mu \Longrightarrow (\phi_1 + \phi_2) \perp \mu$
(3) $\phi_1 \ll \mu$ and $\phi_2 \perp \mu \Longrightarrow \phi_1 \perp \phi_2$
(4) $\phi \ll \mu$ and $\phi \perp \mu \Longrightarrow \phi = 0$

Proof: (1) $\mu(E)=0 \Rightarrow \phi_1(E) = 0$ and $\phi_2(E)=0 \Rightarrow (\phi_1+\phi_2)(E) = \phi_1(E) + \phi_2(E) = 0 \Rightarrow (\phi_1+\phi_2) \ll \mu$

(2) $\left.\begin{array}{l} \phi_1 \text{ concentrated on } A_1 : \phi_1(A_1^c) = 0 \\ \phi_2 \text{ concentrated on } A_2 : \phi_2(A_2^c) = 0 \\ \mu \text{ concentrated on } B : \mu(B^c) = 0 \end{array}\right\}$ we know that : $\left\{\begin{array}{l} A_1 \cap B = \emptyset \\ A_2 \cap B = \emptyset \end{array}\right. \Rightarrow (A_1 \cup A_2) \cap B = \emptyset$

but $\left.\begin{array}{l}(\phi_1+\phi_2) \text{ concentrated on } A_1 \cup A_2 \\ \mu \text{ concentrated on } B\end{array}\right\} \perp$

(3) $\left.\begin{array}{l} \phi_2 \text{ concentrated on } A \\ \mu \text{ concentrated on } B \end{array}\right\} \Rightarrow A \cap B = \emptyset \Rightarrow \mu(A) = 0$ and $\phi_1 \perp \mu \Rightarrow \phi_1(A) = 0$

$\left.\begin{array}{l} \Rightarrow \phi_1 \text{ concentrated on } A^c \\ \text{But } \phi_2 \text{ concentrated on } A \end{array}\right\} \phi_1 \perp \phi_2.$

(4) $\left.\begin{array}{l} \phi \text{ concentrated on } A \\ \mu \text{ concentrated on } B \end{array}\right\} \Rightarrow A \cap B = \emptyset$: $\mu(A) = 0 \Rightarrow \phi(A) = 0$ as $\phi \ll \mu$

ϕ concentrated on $A \Rightarrow \forall E \in \mathcal{F} : \phi(E) = \phi(E \cap A) \leq \phi(A) = 0$

$\Rightarrow \phi(E) = 0 \ \forall E \in \mathcal{F}$.

(II) Theorem :

Lebesgue - Radon - Nikodym thm: Let ϕ and μ be finite measures on $(X, \mathcal{F})$

Then :
- $\exists!$ positive measures ϕ_s and ϕ_c ST : (a) $\phi_c \ll \mu$
  (b) $\phi_s \perp \mu$
  (c) $\phi = \phi_s + \phi_c$
- $\exists$ a measurable function $f : X \to [0, \infty)$ ST : $\phi_c(E) = \int_E f \, d\mu$, $\forall E \in \mathcal{F}$

Note :
- ϕ_c = "continuous part of ϕ WRT μ"
- ϕ_s = "Singular part of ϕ WRT μ"

Radon-Nikodym derivative: If $\phi \ll \mu$, $\exists f : X \to [0, \infty)$ ST $\phi(E) = \int_E f \, d\mu \ \forall E \in \mathcal{F}$

$f := \frac{d\phi}{d\mu}$ is the "Radon-Nikodym derivative of ϕ WRT μ"

Rk: Actually, μ and ϕ only need to be σ-finite

σ-finiteness: μ is σ-finite on $(X, \mathcal{F})$ if : $\exists$ countably many disjoint $(E_n)_{n=1}^{\infty} \in \mathcal{F}$, ST : $\left\{\begin{array}{l} X = \bigcup_{n=1}^{\infty} E_n \\ \mu(E_n) < \infty \end{array}\right.$

Proof of the Lebesgue - Radon - Nikodym theorem:

- Let $\mathcal{A} := \{ g : X \to [0,\infty)$ MF° $\mid \int_E g\,d\mu \le \phi(E), \forall E \in \mathcal{F} \}$ Note: $\mathcal{A} \neq \emptyset$ as $g(x) \equiv 0 \in \mathcal{A}$.
  Let $L := \sup \{ \int_X g\,d\mu : g \in \mathcal{A} \}$: then $0 \le L \le \phi(X) < \infty$ (finite measure)
  Let $(g_n)_{n=1}^{\infty} \in \mathcal{A}$ be a seq of funct° ST: $\lim_{n\to\infty} \int_X g_n\,d\mu = L$
  Let $f_n(x) := \max_{1 \le k \le n} g_k(x)$: $(f_n)_{n=1}^{\infty}$ is $\uparrow$ and positive $\Longrightarrow \lim_{n\to\infty} f_n(x) = f(x)$ exists!

- claim 1: $\int_X f\,d\mu = L$ and $f \in \mathcal{A}$
  $f_n \ge g_n \Rightarrow \int_X f_n\,d\mu \ge \int_X g_n\,d\mu$
  - Monotone Convergence $\Rightarrow \int_X f\,d\mu = \lim_{n\to\infty} \int_X f_n\,d\mu \ge \lim_{n\to\infty} \int_X g_n\,d\mu = L \Rightarrow \int_X f\,d\mu \ge L.$
  - Goal: $f \in \mathcal{A}$ and $\int_X f\,d\mu \le L$
    (∗) Let $A_k := \{ x \in X : f_n(x) = g_k(x) \}, \forall 1 \le k \le n$: So $f_n = \max_{1\le k\le n} g_k(x) \Rightarrow X = \bigcup_{k=1}^{n} A_k$
    Set: $\begin{cases} B_1 = A_1 \\ B_k = A_k \setminus (B_1 \cup \dots \cup B_{k-1}) \end{cases} \Rightarrow B_1, \dots, B_n$ are disjoint and $B_1 \cup \dots \cup B_k = A_1 \cup \dots \cup A_k \; \forall k$, So: $B_1 \cup \dots \cup B_n = X$.
    Let $E \in \mathcal{F}$: $\int_E f_n\,d\mu = \int_E f_n \cdot \underbrace{\left(\sum_{k=1}^{n} \chi_{B_k}\right)}_{=1} d\mu = \sum_{k=1}^{n} \int_E f_n \chi_{B_k}\,d\mu = \sum_{k=1}^{n} \int_{E\cap B_k} f_n\,d\mu \underset{f_n = g_k \text{ on } E\cap B_k}{=} \sum_{k=1}^{n} \int_{E \cap B_k} g_k\,d\mu \le \sum_{k=1}^{n} \phi(E \cap B_k) = \phi(E)$
  So $\forall E \in \mathcal{F}$: $\int_E f_n\,d\mu \le \phi(E) \;\forall n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} \int_E f_n\,d\mu \le \phi(E) \Rightarrow \int_E f\,d\mu \le \phi(E)$, so $f \in \mathcal{A}$

- Set $\phi_c(E) := \int_E f\,d\mu$, $\forall E \in \mathcal{F}$: ϕ_c is a measure on $(X, \mathcal{F})$
  $\phi_c \ll \mu$, and by prev claim $\Rightarrow \phi_c(E) \le \phi(E) \;\forall E \in \mathcal{F}$
  (∗) ϕ, ϕ_c finite & count. additive (∗) ϕ, ϕ_c finite
  Set $\phi_s(E) := \phi(E) - \phi_c(E)$, $\forall E \in \mathcal{F}$: $\phi_s \ge 0$ and ϕ_s is countably additive $\Rightarrow \phi_s$ is a finite measure on $(X, \mathcal{F})$

- claim 2: $\phi_s \perp \mu$, i.e.: ϕ_s and μ are concentrated on disjoint sets
  (∗) $\forall n \in \mathbb{N}$: $\phi_n := \phi_s - \frac{1}{n}\mu$ defined $\forall E \in \mathcal{F} \Rightarrow \phi_n$ is a signed measure on $(X, \mathcal{F})$
  → Jordan decomp°: $\phi_n = \phi_n^+ - \phi_n^-$ and $\exists D_n \in \mathcal{F}$ ST: $\begin{cases} \phi_n^+(E) = \phi_n(E \cap D_n^c) \\ \phi_n^-(E) = -\phi_n(E \cap D_n) \end{cases}$
  Let $D = \bigcap_{n=1}^{\infty} D_n$: $\forall n \in \mathbb{N}$, $\phi_n(D) = \phi_n^+(D) - \phi_n^-(D) = 0 - \phi_n^-(D) \le 0$ (∗) $\phi_n^+(D) = \phi_n(D \cap D_n^c) = \phi_n(\emptyset) = 0$
  So $\phi_n(D) = \phi_s(D) - \frac{1}{n}\mu(D) \le 0 \;\forall n \in \mathbb{N} \Rightarrow 0 \le \phi_s(D) \le \frac{1}{n}\mu(D) \;\forall n \in \mathbb{N} \underset{n\to\infty}{\Longrightarrow} \phi_s(D) = 0 \Rightarrow \phi_s$ concentrated on D^c.
  So ϕ_s concentrated on D^c
  claim: μ concentrated on D, i.e. $\mu(D^c) = 0$
  (∗) $D = \bigcap_{n=1}^{\infty} D_n \Rightarrow D^c = \bigcup_{n=1}^{\infty} D_n^c \Rightarrow \mu(D^c) \le \sum_{n=1}^{\infty} \mu(D_n^c)$ → need to show $\mu(D_n^c) = 0 \;\forall n \in \mathbb{N}$.
  $\int_E (f + \frac{1}{n}\chi_{D_n^c})\,d\mu = \int_E f\,d\mu + \frac{1}{n}\int_E \chi_{D_n^c}\,d\mu = \phi_c(E) + \frac{1}{n}\mu(E \cap D_n^c) = \phi_c(E) + \phi_s(E \cap D_n^c) - \overbrace{\phi_n(E \cap D_n^c)}^{=\phi_n^+(E) \ge 0}$
  $\frac{1}{n}\mu = \phi_s - \phi_n$
  $\le \phi_c(E) + \phi_s(E \cap D_n^c)$
  $\le \phi_c(E) + \phi_s(E) = \phi(E)$
  $\Rightarrow \int_E (f + \frac{1}{n}\chi_{D_n^c})\,d\mu \le \phi(E) \;\forall E \in \mathcal{F} \Rightarrow (f + \frac{1}{n}\chi_{D_n^c}) \in \mathcal{A}$
  $\Rightarrow \int_E (f + \frac{1}{n}\chi_{D_n^c})\,d\mu \le L = \sup_{g \in \mathcal{A}} \int_X g\,d\mu \Rightarrow \underbrace{\int_X f\,d\mu}_{=L} + \frac{1}{n}\mu(D_n^c) \le L \Rightarrow 0 \le \mu(D_n^c) \le 0 \Rightarrow \mu(D_n^c) = 0 \;\forall n$
  So: $\mu \perp \phi_s$

- Uniqueness: Suppose $\phi = \phi_c^{(1)} + \phi_s^{(1)} = \phi_c^{(2)} + \phi_s^{(2)}$ where $\begin{cases} \phi_s^{(1)} \perp \mu \\ \phi_c^{(1)} \ll \mu \end{cases}$ and $\begin{cases} \phi_s^{(2)} \perp \mu \\ \phi_c^{(2)} \ll \mu \end{cases}$
  Suppose: • μ concentrated on A_1 and $\phi_s^{(1)}(A_1) = 0$
  • μ concentrated on A_2 and $\phi_s^{(2)}(A_2) = 0$
  Let $A = A_1 \cap A_2$: $0 \le \mu(A^c) \le \mu(A_1^c) + \mu(A_2^c) = 0$
  $\Rightarrow \mu(A^c) = 0$: μ is concentrated on A.

μ concentrated on $A \Rightarrow \phi_s^{(1)}(A) = \phi_s^{(2)}(A) = 0 \to \phi_s^{(1)}(A \cap E) = \phi_s^{(2)}(A \cap E) = 0 \;\forall E \in \mathcal{F}$
$\begin{cases} \phi_c^{(1)} \ll \mu \Rightarrow \phi_c^{(1)} \text{ concentrated on } A \\ \phi_c^{(2)} \ll \mu \Rightarrow \phi_c^{(2)} \text{ concentrated on } A \end{cases}$
$\Rightarrow \forall E \in \mathcal{F}$: $\phi_c^{(1)}(E) - \phi_c^{(2)}(E) = \phi_c^{(1)}(E \cap A) - \phi_c^{(2)}(E \cap A) = \underbrace{\phi_s^{(1)}(E \cap A)}_{=0} - \underbrace{\phi_s^{(2)}(E \cap A)}_{=0}$
$\Rightarrow \forall E \in \mathcal{F}$: $\phi_c^{(1)}(E) = \phi_c^{(2)}(E)$
$\Rightarrow \forall E \in \mathcal{F}$: $\phi_s^{(1)}(E) = \phi_s^{(2)}(E)$ (∗) $\phi_c^{(1)} + \phi_s^{(1)} = \phi_c^{(2)} + \phi_s^{(2)}$ by assumpt°

Conditional Expectation 8

① Conditional Expectation:

$(X, \mathcal{F}, \mu)$ a probability space : $\mu(X) = 1$

Random Variable: measurable function $f : X \to \mathbb{R}$

Expectation of a r.v.: If $\int_X |f| d\mu < \infty$, then: $\mathbb{E}(f) := \int_X f \, d\mu$

Conditional expectation: Let $\mathcal{A} \subset \mathcal{F}$ be a σ-Algebra.
Let $\phi(E) := \int_E f \, d\mu$, $\forall E \in \mathcal{A}$ $\to$ ϕ is a signed measure on $(X, \mathcal{A})$

LRN thm $\Rightarrow$ $\mathbb{E}(f|\mathcal{A}) := \frac{d\phi}{d\mu}$ with μ restricted to $(X, \mathcal{A})$

(OR) ϕ measure on $(X, \mathcal{A})$
μ probability measure on $(X, \mathcal{A})$
and $\phi \ll \mu|_{\mathcal{A}}$

thm: $\mathbb{E}(f|\mathcal{A})$ is a function that is uniquely specified by:

(1) $\mathbb{E}(f|\mathcal{A})$ is $\mathcal{A}$-measurable

(2) $\forall E \in \mathcal{A}$: $\int_E \mathbb{E}(f|\mathcal{A}) d\mu = \int_E f \, d\mu = \phi(E)$

Rk: f may not be measurable WRT $\mathcal{A}$!

Prop: f measurable WRT $\mathcal{A} \Rightarrow \mathbb{E}(f|\mathcal{A}) = f$

Prop: (1) Linearity : $\mathbb{E}(\alpha f + \beta g|\mathcal{A}) = \alpha \mathbb{E}(f|\mathcal{A}) + \beta \mathbb{E}(g|\mathcal{A})$

(2) Let $\mathcal{A} = \{\phi, X\}$: then $\mathbb{E}(f|\mathcal{A}) = \mathbb{E}(f) = \int_X f \, d\mu$

Proof: (1) Linearity of integral

(2) If $g : X \to \mathbb{R}$ is $\mathcal{A}$-measurable, then $g = CST$ (we will take $g = \mathbb{E}(f|\mathcal{A})$)

(OR) if $x, y \in X$: if $g(x) \neq g(y)$, take r ST $g(x) < r < g(y)$
then $g^{-1}((-\infty, r)) \neq \phi$ and $g^{-1}((r, \infty)) \neq \phi$ are disjoint & measurable $\boxed{\Rightarrow\Leftarrow}$ as $\mathcal{A} = \{\phi, X\}$

So $\mathbb{E}(f|\mathcal{A}) = CST$

$\Rightarrow \int_X \mathbb{E}(f|\mathcal{A}) d\mu = \mathbb{E}(f|\mathcal{A}) \int_X d\mu = \mathbb{E}(f|\mathcal{A}) \cdot \underbrace{\mu(X)}_{=1} = \mathbb{E}(f|\mathcal{A}) = \int_X f \, d\mu$ $\Rightarrow \mathbb{E}(f|\mathcal{A}) = \mathbb{E}(f)$

Example: $\mathcal{A} = \{\phi, E, E^c, X\}$ where $\begin{cases} \mu(E) > 0 \\ \mu(E^c) > 0 \end{cases}$

Prop: $\mathbb{E}(f|\mathcal{A}) = \alpha \chi_E + \beta \chi_{E^c}$, where $\begin{cases} \alpha = \frac{1}{\mu(E)} \int_E f \, d\mu \to \mu_E(A) \\ \beta = \frac{1}{\mu(E^c)} \int_{E^c} f \, d\mu \to \mu_{E^c}(A) \end{cases}$ Probability measures in $(X, \mathcal{F})$

Proof: • $g : X \to \mathbb{R}$ measurable WRT $\mathcal{A}$: then $g = \alpha \chi_E + \beta \chi_{E^c}$ (take $g = \mathbb{E}(f|\mathcal{A})$)

(OR) $g = CST$ on E : if $g(x) \neq g(y)$, take r st $g(x) < r < g(y)$ $\Rightarrow \underbrace{g^{-1}((-\infty, r))}_{\in \mathcal{A}}$ & $\underbrace{g^{-1}((r, \infty))}_{\in \mathcal{A}}$ nonempty & disjoint & measurable $\boxed{\Rightarrow\Leftarrow}$

$g = CST$ on E^c : Same argument

• $\mathbb{E}(f|\mathcal{A}) = \alpha \chi_E + \beta \chi_{E^c}$: So $\int_E f \, d\mu = \int_E \mathbb{E}(f|\mathcal{A}) = \alpha \mu(E) \Rightarrow \alpha = \frac{1}{\mu(E)} \int_E f \, d\mu$.
(Same for β)

Note: • $\frac{1}{\mu(E)} \int_E f \, d\mu = \int_X f \, d\mu_E$ where $\mu_E(A) = \frac{\mu(A \cap E)}{\mu(E)}$ $\to$ Conditional prob that A happens given that E happens.

• $\frac{1}{\mu(E^c)} \int_{E^c} f \, d\mu = \int_X f \, d\mu_{E^c}$ where $\mu_{E^c}(A) = \frac{\mu(A \cap E^c)}{\mu(E^c)}$ $\to$ conditional prob that A happens given that E did not happen.

IV Expectation:

$(X, \mathcal{F}, \mu)$ a probability space. Let $f: X \to \mathbb{R}$ be a r.v.

Expectation: of a r.v. f: $\mathbb{E}(f) = \int_X f \, d\mu$

Random process: $\{f_n\}_{n=1}^{\infty}$ a seq of random variables (i.e. MF°)

$\{f_n(x)\}_{n=1}^{\infty} = (f_1(x), \dots, f_n(x), \dots)$ the sequence of realisations for any given $x \in X$.

Average value of the outcome: knowing that $x \in X$ has occurred: $S_n(x) = \frac{f_1(x) + \dots + f_n(x)}{n}$

Independent r.v.: $\forall n_1, \dots, n_k \in \mathbb{N}$ distinct, and $\forall B_{n_1}, \dots, B_{n_k} \subset \mathbb{R}$ Borel:

$$\mu(f_{n_1}^{-1}(B_{n_1}) \cap \dots \cap f_{n_k}^{-1}(B_{n_k})) = \mu(f_{n_1}^{-1}(B_{n_1})) \cdot \ldots \cdot \mu(f_{n_k}^{-1}(B_{n_k}))$$

$\Rightarrow f_{n_i}^{-1}(B_{n_i})$ are indep events.

Identically distributed r.v.: $\mu_{f_n} = \mu_{f_m} \; \forall n, m$ (where: $\mu_{f_n}(B) := \mu(f_n^{-1}(B))$)

$\rightarrow$ a repetition of the same experiment

Kolmogorov's law of large nb: Let $\{f_n\}_{n=1}^{\infty}$ be a seq of IID r.v.

Then for μ-almost all $x \in X$: $\lim_{n \to \infty} S_n(x) = \underbrace{\int_X f \, d\mu}_{\text{does not depend on the } x \text{ chosen}}$

Note: This is why we define $\mathbb{E}(f) := \int_X f \, d\mu$

Generalizat°: Birkhoff ergodic theorem

Fundamental theorem of Calculus:

Fund. thm. of Calc: $f : [a,b] \to \mathbb{R}$ continuous, $F(x) := \int_a^x f(t)\,dt$

(1) F is differentiable on (a,b) and $F'(x) = f(x)$ $\forall x \in (a,b)$

(2) if $f'(x)$ $\exists$ and is continuous on $[a,b]$: $f(x) - f(a) = \int_a^x f'(t)\,dt$

Maximal functions: $f : \mathbb{R} \to \mathbb{C}$ measurable, $\int_{\mathbb{R}} |f|\,dm < \infty$, $I(x,r) := (x-r, x+r) \to m(I(x,r)) = 2r$.

↳ Lebesgue Measure!

$$(Mf)(x) := \sup_{r \to 0} \left(\frac{\int_{I(x,r)} |f|\,dm}{m(I(x,r))} \right)$$

So: $$(Mf)(x) = \sup_{r \to 0} \frac{1}{2r} \int_{[x-r,x+r]} |f|\,dm$$

Minimal function: $(X, \mathcal{F}, \mu)$, $f : X \to \mathbb{R}$ integrable, $\mu(X) = 1$:

$$\int_X f\,d\mu = a, \quad \int_X f^2\,d\mu = a^2 \implies f(x) = a \ \ \mu\text{-almost everywhere}$$

(pf) $\int_X (f-a)^2\,dm = \int_X f^2\,dm - 2a\int_X f\,dm + a^2 = a^2 - 2a^2 + a^2 = 0$

$\Rightarrow (f-a) = 0$ almost everywhere.

Prop: $\forall \lambda \in \mathbb{R}$: $\{x \in \mathbb{R} : (Mf)(x) > \lambda\}$ is open & Mf is measurable

Vitali Covering lemma: W a set in $\mathbb{R}$, $I(x_j, r_j)$, $j = 1, \dots, N$ intervals ST: $W \subset \bigcup_{j=1}^N I(x_j, r_j)$

Then: $\exists S \subset \{1, \dots, N\}$, ST $\{I(x_j, r_j)\}_{j \in S}$ are disjoint intervals, and $W \subset \bigcup_{j \in S} I(x_j, 3r_j)$

Prop: $\forall \lambda > 0$: $m(\{x \in \mathbb{R} : (Mf)(x) > \lambda\}) \leq \frac{3}{\lambda} \int_{\mathbb{R}} |f|\,dm$

Lebesgue point of f: $x \in \mathbb{R}$: • f continuous at x

and • $\lim_{r \to 0} \frac{1}{m(I(x,r))} \int_{I(x,r)} |f(y) - f(x)|\,dm(y) = 0$

Thm: f is integrable $\Rightarrow$ m-almost every $x \in \mathbb{R}$ are Lebesgue points of f.

↳ $m(\{x : x \text{ is not a Lebesgue pt of } f\}) = 0$

Exercise: Given $n > 0$: $\exists g \in C_c(\mathbb{R})$, ST: $\int_{\mathbb{R}} |f - g|\,dm < \frac{1}{n}$

(pf) • $S = \sum_{k=1}^m r_k \chi_{E_k}$, $\mu(E_k) < \infty$, $r_k \in \mathbb{R}$ with $\int_{\mathbb{R}} |f - s|\,dm < \frac{1}{2n}$

• Apply Lusin's thm to construct $g \in C_c(\mathbb{R})$ ST: $\int_{\mathbb{R}} |g - s|\,dm < \frac{1}{2n}$

Fund thm of Calculus: • $f : \mathbb{R} \to \mathbb{R}$ integrable

• $F(x) = \int_{(-\infty, x]} f(y)\,dm(y) = \int_{-\infty}^x f(y)\,dy$ is a continuous function

Then: if $x \in \mathbb{R}$ is a Lebesgue pt of F: then F is differentiable at x: $F'(x) = f(x)$.