Order Statistics:

- $Y_i = i^{th}$ min of $\{X_1, \dots, X_n\}$: $Y_1 \le Y_2 \le \dots \le Y_n$ (strict for X_i cont r.v.)

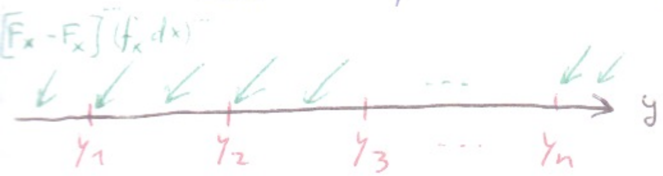
(Note: $X_1, \dots, X_n$ are IID r.v.)
⚠ $Y_1, \dots, Y_n$ are NOT IID)

Thm: $X_1, \dots, X_n$ cont IID r.v. with PDF $f_X(x)$:

$$f_{Y_1,\dots,Y_n}(y_1, \dots, y_n) = \begin{cases} n! \prod_{i=1}^{n} f_X(y_i) & , \text{if } y_1 < y_2 < \dots < y_n \\ 0 & , \text{o'wise} \end{cases}$$

- Multinomial dist:
  - → n indep trials that can result in 1 of K outcomes
  - → p_i = prob(outcome i) at any trial
  - → X_i = # of trials resulting in outcome i

$$\Rightarrow \begin{cases} P(X_1 = n_1, \dots, X_K = n_K) = \frac{n!}{n_1! \cdots n_K!} p_1^{n_1} \cdots p_K^{n_K} \\ \sum_{i=1}^{K} n_i = n \\ \sum_{i=1}^{K} p_i = 1 \end{cases}$$



- Transformation theorem: $\begin{cases} \tilde{X} = (X_1, \dots, X_n) \\ \tilde{Y} = (Y_1, \dots, Y_n) \end{cases}$ $\quad \tilde{g}: \tilde{X} \longrightarrow \tilde{Y}$, $\quad \tilde{x} \longmapsto \tilde{y} = (g_1(\tilde{x}), \dots, g_n(\tilde{x}))$

$$\boxed{f_{\tilde{Y}}(\tilde{y}) = \sum_{\ell=1}^{L} f_{\tilde{X}}(\tilde{g}_\ell^{-1}(\tilde{y})) \cdot |J_\ell|}$$

where $\begin{cases} \tilde{g}_\ell = \tilde{g}|_{R_\ell} & \text{where } g \text{ is 1-1} \\ J_\ell = \det\left(\frac{\partial x_i}{\partial y_j} \Big|_{x \in R_\ell} \right) \end{cases}$

Convergence thms:

X_i IID!

- Sample mean: $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$
- Sample variance: $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$

WLLN: $E(X_i) = \mu < \infty$, X_i IID:

$$\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow[n \to \infty]{P} \mu = E(X_i)$$

- $X_1, \dots, X_n$ IID $N(\mu, \sigma^2)$: $\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$

- **CLT:** $\left. \begin{matrix} \frac{\bar{X}_n - \mu}{\sigma / \sqrt{n}} \\ \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}} \end{matrix} \right\} \xrightarrow[n \to \infty]{D} N(0,1)$

Note: $X \xrightarrow{P} Y \Rightarrow X \xrightarrow{D} Y$

- **Slutski:** $\left. \begin{matrix} X_n \xrightarrow{D} X \\ Y_n \xrightarrow{P} c \in \mathbb{R} \end{matrix} \right\} \Rightarrow \begin{matrix} X_n \pm Y_n \xrightarrow{d} X \pm c \\ X_n Y_n \xrightarrow{d} cX \\ \frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{c} \end{matrix}$

- Continuous mapping thm: g a cont. function
  - $X_n \xrightarrow{D} X \Rightarrow g(X_n) \xrightarrow{D} g(X)$
  - $X_n \xrightarrow{P} X \Rightarrow g(X_n) \xrightarrow{P} g(X)$

- $\frac{\bar{X}_n - \mu}{S / \sqrt{n}} \xrightarrow[n \to \infty]{D} N(0,1)$

- $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{D} X$
- $X_n \xrightarrow{P} c \Leftrightarrow X_n \xrightarrow{D} c$, $c \in \mathbb{R}$ CST

- $\left. \begin{matrix} X_n \xrightarrow{D} X \\ Y_n \xrightarrow{D} Y \\ X_n \perp\!\!\!\perp Y_n, \ X \perp\!\!\!\perp Y \end{matrix} \right\} \Rightarrow X_n + Y_n \xrightarrow{D} X + Y$

⚠ $X_n Y_n \cancel{\xrightarrow{D}} XY$ in general

Distributions:

- $Z \sim N(0,1) \Rightarrow Z^2 \sim \chi^2_1$
- $X_i \sim \chi^2_{\nu_i}$ $\underline{\underline{IID}} \longrightarrow \sum_{i=1}^{n} X_i \sim \chi^2_{\sum_{i=1}^{n} \nu_i}$
- $\left.\begin{matrix} Y \sim \chi^2_\nu \\ Z \sim N(0,1) \end{matrix}\right\} Y \perp\!\!\!\perp Z \Rightarrow \dfrac{Z}{\sqrt{Y/\nu}} \sim t_\nu$

$$\boxed{[N(0,1)]^2 = \chi^2_1}$$

$$\boxed{\sum \chi^2_{\nu_i} = \chi^2_{\sum \nu_i}}$$

$$\boxed{\frac{N(0,1)}{\sqrt{\chi^2_\nu / \nu}} = t_\nu}$$

- $X_1, \dots, X_n$ $\underline{IID}$ $N(\mu, \sigma^2)$: $\bar{X} \perp\!\!\!\perp S^2$ and $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

Note: $X \perp\!\!\!\perp Y$, g cont $\Rightarrow X \perp\!\!\!\perp g(Y)$

- $X_1, \dots, X_n$ $\underline{IID}$ $N(\mu, \sigma^2)$: $\dfrac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$

- $\left.\begin{matrix} X \sim \chi^2_{\nu_1} \\ Y \sim \chi^2_{\nu_2} \end{matrix}\right\} X \perp\!\!\!\perp Y \Rightarrow \dfrac{X/\nu_1}{Y/\nu_2} \sim F_{\nu_1, \nu_2}$

$$\boxed{\frac{\chi^2_{\nu_1}/\nu_1}{\chi^2_{\nu_2}/\nu_2} \sim F_{\nu_1, \nu_2}}$$

- $\left.\begin{matrix} (X_1, \dots, X_{n_1}) \perp\!\!\!\perp (Y_1, \dots, Y_{n_2}) \\ X_i \sim N(\mu_1, \sigma^2) \ \underline{IID} \\ Y_i \sim N(\mu_2, \sigma^2) \ \underline{IID} \end{matrix}\right\} \dfrac{S_1^2}{S_2^2} \sim F_{n_1 - 1, n_2 - 1}$

- $X_i \sim \text{Poisson}(\lambda_i)$ $\underline{IID}$ $\Rightarrow \sum_{i=1}^{n} X_i \sim \text{Poisson}\left(\sum_{i=1}^{n} \lambda_i\right)$

$$\boxed{\sum \text{Poisson}(\lambda_i) = \text{Poisson}(\sum \lambda_i)}$$

Table of Discrete and Continuous distributions

$E[X^n] = M^{(n)}(0)$

$M(t) = E[e^{tX}]$, $M(0) = 1$

$Var[X] = E[X^2] - (E[X])^2 = E[(X-\mu)^2]$

Distribution	Type	Mass/density function $f(x)$ (PMF / PDF)	Mean μ ($E[X]$)	Variance σ^2	Moment Generating Function
UNIFORM(n)	D	$1/n$, for $x = 1, 2, \ldots, n$	$(n+1)/2$	$(n^2-1)/12$	$\frac{e^{at} - e^{(b+1)t}}{(b-a+1)(1-e^t)}$, $t \in \mathbb{R}$
UNIFORM(a, b)	C	$\frac{1}{b-a}$, for $x \in [a, b]$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{tb} - e^{ta}}{tb - ta}$, $t \in \mathbb{R}$
BERNOULLI(p)	D	$f(0) = 1-p$, $f(1) = p$	p	$p(1-p)$	$q + pe^t$, $t \in \mathbb{R}$
BINOMIAL(n, p)	D	$\binom{n}{x} p^x (1-p)^{n-x}$, for $x = 0, 1, \ldots, n$	np	npq	$(q + pe^t)^n$, $t \in \mathbb{R}$
GEOMETRIC(p)	D	$q^{x-1}p$, for $x = 1, 2, \ldots$	$1/p$	$(1-p)/p^2$	$\frac{p}{1-qe^t}$, $t < -\log q$
POISSON(λt)	D	$\frac{1}{x!}(\lambda t)^x e^{-\lambda t}$, for $x = 0, 1, \ldots$	λt	λt	$e^{\lambda(e^t - 1)}$, $t \in \mathbb{R}$
HYPERGEOMETRIC(N, M, n)	D	$\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$, for $x = 0, 1, \ldots, n$	np	$np(1-p)$	too complex
EXPONENTIAL(λ)	C	$\lambda e^{-\lambda x}$, for $x \in [0, \infty)$	$1/\lambda$	$1/\lambda^2$	$\frac{\lambda}{\lambda - t}$, $t < \lambda$
GAMMA(α, β)	C	$\frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}$, for $x \in [0, \infty)$	$\alpha\beta$	$\alpha\beta^2$	$\left(\frac{1}{1-\beta t}\right)^\alpha$, $t < \frac{1}{\beta}$
NORMAL(μ, σ^2)	C	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$, for $x \in \mathbb{R}$	μ	σ^2	$e^{\mu t + \frac{\sigma^2 t^2}{2}}$, $t \in \mathbb{R}$
Degenerate (a)	D	$P(X = x) = \begin{cases} 1 & x = a \\ 0 & x \neq a \end{cases}$	a	0	e^{at}, $t \in \mathbb{R}$
Multinomial $(m, p_1, \ldots, p_{k-1})$		$P(X_1 = x_1, \ldots, X_k = x_k) = \binom{m}{x_1, \ldots, x_k} \prod_{j=1}^{k} p_j^{x_j}$, $\binom{m}{x_1, \ldots, x_k} = \frac{m!}{x_1! \cdots x_k!}$			

$N(0,1)$: $E(X^n) = \begin{cases} 0 & n \text{ even} \\ (n-1)(n-3)\cdots 3 \cdot 1 & n \text{ odd} \end{cases}$

- $\text{Gamma}(\alpha,\beta)$: $\dfrac{x^{\alpha-1}\, e^{-x/\beta}}{\beta^{\alpha}\,\Gamma(\alpha)}$ $\forall x \geq 0$: $\mu = \alpha\beta$, $\sigma^2 = \alpha\beta^2$, $M(t) = \left(\dfrac{1}{1-\beta t}\right)^{\alpha}$ $\forall t < \frac{1}{\beta}$

  - $\text{Exp}(\beta)$: $\dfrac{1}{\beta}\, e^{-x/\beta}$ $\forall x \geq 0$: $\mu = \beta$, $\sigma^2 = \beta^2$, $M(t) = \dfrac{1}{1-\beta t}$ $\forall t < \frac{1}{\beta}$

    $= \text{Gamma}(1,\beta)$

  - χ^2_ν : $\dfrac{x^{\frac{\nu}{2}-1}\, e^{-x/2}}{2^{\nu/2}\,\Gamma\left(\frac{\nu}{2}\right)}$ $\forall x \geq 0$: $\mu = \nu$, $\sigma^2 = 2\nu$, $M(t) = \left(\dfrac{1}{1-2t}\right)^{\frac{\nu}{2}}$ $\forall t < \frac{1}{\beta}$

    $= \text{Gamma}\left(\frac{\nu}{2}, 2\right)$

- Student T_ν : $\dfrac{\Gamma\left(\frac{1+\nu}{2}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)} \cdot \left(1+\dfrac{x^2}{\nu}\right)^{-\frac{1+\nu}{2}}$, $x \in \mathbb{R}$: $\underset{\nu>1}{\mu = 0}$, $\underset{\nu>2}{\sigma^2 = \dfrac{\nu}{\nu-2}}$

  has moments of order $< \nu$

- $F(\nu_1,\nu_2)$: $\dfrac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \cdot \left(\dfrac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} \cdot \dfrac{x^{\frac{\nu_1}{2}-1}}{\left(1+x\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1+\nu_2}{2}}}$, $\forall x \geq 0$: $\underset{\nu_2>2}{\mu = \dfrac{\nu_2}{\nu_2-2}}$, $\underset{\nu_2>4}{\sigma^2 = \dfrac{2\nu_2^2}{\nu_1} \cdot \dfrac{\nu_1+\nu_2-2}{(\nu_2-2)^2(\nu_2-4)}}$

Estimators:

- Statistic = measurable function $T(X_1, \ldots, X_n)$
- estimator = a Statistic $\hat{\theta} = \hat{\theta}(X_1, \ldots, X_n)$ is an estimator for θ if $\hat{\theta}$ does not depend on θ functionally or on any unknown parameters

Goal = Find "recipies" / "procedures" that work well on average when the procedure is used repeatedly

⚠ We can never say how well we have done on any particular occasion!

Criteria of "goodness" that estimators should have:

1 Unbiasedness:

Def: $X_1, \ldots, X_n$ IID depend on θ unknown : $\hat{\theta}$ unbiased $\Rightarrow E_\theta(\hat{\theta}) = \theta, \forall \theta \in \Theta$.

Bias: $b(\hat{\theta}) = E_\theta(\hat{\theta}) - \theta$

2 UMVUEs: (Uniform Minimum Variance Unbiased Estimators)

Def: $\hat{\theta}$ UMVUE for $\theta \Leftrightarrow \begin{cases} E_\theta(\hat{\theta}) = \theta, \forall \theta \in \Theta \\ \text{if } \hat{\theta}^* \text{ is another unbiased estimator: } Var(\hat{\theta}^*) \geq Var(\hat{\theta}), \forall \theta \in \Theta \end{cases}$

⚠ UMVUES are unique if they exist!

Cramér-Rao ineq = $\tilde{X} = (X_1, \ldots, X_n)$ with PDF (PMF) $F_{\tilde{X}}(\tilde{x}; \theta)$, $E[T(\tilde{X})] = g(\theta)$ for a stat T, g diff.

$$\Rightarrow \boxed{Var[T(\tilde{X})] \geq \frac{(g'(\theta))^2}{E\left[\left(\frac{\partial}{\partial\theta} \ln f_{\tilde{X}}(\tilde{X}, \theta)\right)^2\right]} \overset{\text{IID}}{=} \frac{(g'(\theta))^2}{nE\left[\left(\frac{\partial}{\partial\theta} \ln f_X(X, \theta)\right)^2\right]} = \frac{(g'(\theta))^2}{-n \cdot E\left[\frac{\partial^2}{\partial\theta^2} \ln f_X(X, \theta)\right]}}$$

⚠ a UMVUE does not always reach the CR Lower bound

Mean Square Error: $MSE(\hat{\theta}) = E[(\hat{\theta} - \theta)^2] = Var(\hat{\theta}) + [b(\hat{\theta})]^2$

3 Consistency:

Def: $\hat{\theta}$ Consistent for $\theta \Leftrightarrow \hat{\theta} \xrightarrow[n\to\infty]{P} \theta$, ie: $P(|\hat{\theta} - \theta| > \varepsilon) \xrightarrow[n\to\infty]{} 0 \ \forall \varepsilon > 0$

$P(|\hat{\theta} - \theta| < \varepsilon) \xrightarrow[n\to\infty]{} 1 \ \forall \varepsilon > 0$

methods: (1) WLLN: $X_1, \ldots, X_n$ IID, $\mu < \infty \Rightarrow \frac{1}{n} \sum_{i=1}^n X_i \xrightarrow[n\to\infty]{P} \mu \Rightarrow g(\bar{X}) \xrightarrow[n\to\infty]{P} g(\mu)$

if $g(\mu) = \theta$ then $\hat{\theta} = g(\bar{X})$ is consistent for θ.

(2) $\left.\begin{array}{l} E(\hat{\theta}_n) \xrightarrow[n\to\infty]{} \theta \\ Var(\hat{\theta}_n) \xrightarrow[n\to\infty]{} 0 \end{array}\right\} \Rightarrow \hat{\theta}_n \xrightarrow[n\to\infty]{P} \theta$

④ Sufficiency:

Def: T is sufficient for θ : $f_{\tilde{X}|T}(\tilde{x};\theta \mid T(\tilde{X})=t)$ indep of θ $\underline{\underline{\forall \tilde{x}}}$, $\underline{\underline{\forall t}}$

- $\tilde{T}(\tilde{X}) = \tilde{X}$ always sufficient for θ
- g 1 to 1 : T Sufficient $\Rightarrow g(T)$ sufficient

Neymann-Fisher thm: T is Sufficient $\Longleftrightarrow f_{\tilde{X}}(\tilde{x};\theta) = g(T(\tilde{x});\theta)\, h(\tilde{x})$

min Suff: T is min Suff $\Longleftrightarrow \begin{cases} T \text{ suff for } \theta \\ \forall T' \text{ suff}, \exists g \text{ ST } T = g(T') \end{cases}$

Thm: If " $\dfrac{f_{\tilde{X}}(\tilde{x};\theta)}{f_{\tilde{X}}(\tilde{y};\theta)}$ indep of $\theta \Leftrightarrow T(\tilde{x}) = T(\tilde{y})$ ", then T is min Suff.

(Lemma: T sufficient : $T(\tilde{x}) = T(\tilde{y}) \Rightarrow \dfrac{f_{\tilde{X}}(\tilde{x};\theta)}{f_{\tilde{X}}(\tilde{y};\theta)}$ indep of θ.)

⑤ Completeness:

complete family: a family $\{f_{\tilde{X}}(\tilde{x};\theta) : \theta \in \Theta\}$ is complete $\Longleftrightarrow$ " $E_\theta[g(\tilde{X})] \overset{\theta}{=} 0 \Rightarrow g(\tilde{X}) = 0$ with prob 1" holds $\forall g$ measurable.

complete statistic: T is complete if : " $E_\theta[g(T(\tilde{X}))] \overset{\theta}{=} 0 \Rightarrow g(T(\tilde{X})) = 0$ with prob 1" holds $\forall g$ meas.

Rao-Blackwell improvement thm: $\hat{\theta}$ unbiased for θ, T sufficient for θ.

then $T' = E[\hat{\theta} \mid T]$ is unbiased for θ, and $Var_\theta(T') \le Var_\theta(T)$

Note: if $\hat{\theta}$ not unbiased: $MSE(\hat{\theta}) \ge MSE(T')$

Lehmann-Scheffé thm: $\hat{\theta}$ unbiased for θ, T complete & Sufficient for θ

if $\exists g$ ST $\hat{\theta} = g(T)$, then $\hat{\theta}$ is the unique UMVUE for θ.

How to use:
- find $\hat{\theta}$ unbiased, ST $\hat{\theta} = g(T)$ where T is complete & Suff.

OR
- $\left.\begin{array}{l} \hat{\theta} \text{ unbiased} \\ T \text{ complete \& Suff} \end{array}\right\}$ Take $T' = E[\hat{\theta} \mid T] \longrightarrow$ UMVUE for θ.

EXAMPLES:

Family:	Estimator:	Unbiased:	UMVUE:	Consistent:
$X_1,\dots,X_n$ IID, $\mu<\infty$, $\sigma^2<\infty$	$\hat\mu = \bar X$	✓	✓	✓
	$\hat\sigma^2 = \frac{1}{n}\sum(X_i-\mu)^2$ ↳ known	✓	depends on family	✓
	$\hat S^2 = \frac{1}{n-1}\sum(X_i-\bar X)^2$ ↳ $\mu = ?$	✓	depends on family	✓
$X_1,\dots,X_n$ IID $N(\mu,\sigma^2)$	$\hat\mu = \bar X$	✓	✓	✓
	$\hat\sigma^2 = \frac{1}{n}\sum(X_i-\mu)^2$	✓	✓	✓
	$\hat S^2 = \frac{1}{n-1}\sum(X_i-\bar X)^2$	✓	✓	✓
$X_1,\dots,X_n$ IID Bernoulli(p)	$\hat p = \frac{1}{n}\sum X_i$	✓	✓	✓
$X_1,\dots,X_n$ IID $U(0,\theta)$	$\hat\theta = \frac{n+1}{n}Y_n$	✓	✓	✓
	$\hat\theta = 2\bar X$	✓	✗	✓
$X_1,\dots,X_n$ IID Exp(β), $\mu=\beta$, $\sigma^2=\beta^2$	$\hat\mu = \bar X$	✓	✓	✓
	$\hat\sigma^2 = \bar X^2$	$n\to\infty$	✓	✓
	$\hat S^2 = \frac{1}{n-1}\sum(X_i-\bar X)^2$	✓	✗	✓
	$\hat\theta = e^{-10/\bar X}$ $(\theta = e^{-10/\beta} = P(X_1>10))$	$n\to\infty$		✓
$X_1,\dots,X_n$ IID Poisson(λ)	$\hat\lambda = \bar X$	✓	✓	✓
	$\hat\theta = (1-\frac{1}{n})^{\sum X_i}$ $(\theta = e^{-\lambda} = P(X=0))$	✓	✓	✓

Examples:

Family:	Statistic	Sufficient	min Suff	Complete
$X_1,\dots,X_n$ IID	$T(\tilde{X}) = \tilde{X}$	✓		
$\mu < \infty$, $\sigma^2 < \infty$	$g(T)$ $\forall T$ suff $\forall g$ cont			
$X_1,\dots,X_n$ IID	$T(\tilde{X}) = \sum (X_i - \mu)^2$ (σ^2)	✓	✓	
$N(\mu, \sigma^2)$	$T(\tilde{X}) = \sum X_i$ (μ)	✓	✓	
$X_1,\dots,X_n$ IID	$T(\tilde{X}) = \sum X_i$ (p)	✓		
Bernoulli (p)	$T(\tilde{X}) = \frac{1}{n}\sum X_i$ (p)	✓		
$X_1,\dots,X_n$ IID	$T(\tilde{X}) = \bar{X}$ (θ)	✗	✗	
$U(0,\theta)$	$T(\tilde{X}) = Y_n$ (θ)	✓		✓
$X_1,\dots,X_n$ IID	$T(\tilde{X}) = \prod X_i$ (α)	✓		
Gamma (α, β)	$T(\tilde{X}) = \sum X_i$ (β)	✓		
	$\tilde{T}(\tilde{X}) = (\prod X_i, \sum X_i)$ (α, β)	✓		
$X_1,\dots,X_n$ IID, Poisson (λ)	$T(\tilde{X}) = \sum X_i$	✓	✓	✓

Complete families:

- $X \sim N(\mu, \sigma^2)$, $\mu = ?$, σ^2 known
- $X \sim$ Poisson(λ)

	Population(s)	To test	calculate	and reject H_0 at level α if	$(1-\alpha)\times 100\%$ **Confidence Interval**
1.	$N(\mu,\sigma^2)$ (or *any* population if n is large (≥ 25)) with σ^2 known.	$H_0: \mu = \mu_0$ $H_1: \mu \neq \mu_0$ $(\mu > \mu_0)$ $(\mu < \mu_0)$	$z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$ If n is large, then σ may be replaced by s.	$\lvert z\rvert > z_{\alpha/2}$ $(z > z_\alpha)$ $(z < -z_\alpha)$	$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ If n is large, then σ may be replaced by s.
2.	$N(\mu,\sigma^2)$ with both μ and σ^2 unknown	same as above	$t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$	$\lvert t\rvert > t_{\alpha/2,n-1}$ $(t > t_{\alpha,n-1})$ $(t < -t_{\alpha,n-1})$	$\bar{x} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}$
3.	Bin(p) (large sample case($n \geq 25$)). $\hat{p} = x/n, \hat{q} = 1-\hat{p}$	$H_0: p = p_0$ $H_1: p \neq p_0$ $(p > p_0)$ $(p < p_0)$	$z = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0 q_0}{n}}}$	$\lvert z\rvert > z_{\alpha/2}$ $(z > z_\alpha)$ $(z < -z_\alpha)$	$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}} < p < \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}}$
4.	$N(\mu,\sigma^2)$ with both μ and σ^2 unknown	$H_0: \sigma^2 = \sigma_0^2$ $H_1: \sigma^2 \neq \sigma_0^2$ $(\sigma^2 > \sigma_0^2)$ $(\sigma^2 < \sigma_0^2)$	$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$	$\chi^2 > \chi^2_{\alpha/2,n-1}$ or $< \chi^2_{1-\alpha/2,n-1}$ $(\chi^2 > \chi^2_{\alpha,n-1})$ $(\chi^2 < \chi^2_{1-\alpha,n-1})$	$\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}$
5.	$N(\mu_1,\sigma_1^2)$ and $N(\mu_2,\sigma_2^2)$ (or *any* populations provided n_1 and n_2 are large (≥ 25)), where σ_1^2 and σ_2^2 are known.	$H_0: \mu_1-\mu_2 = D_0$ $H_1: \mu_1-\mu_2 \neq D_0$ $(\mu_1-\mu_2 > D_0)$ $(\mu_1-\mu_2 < D_0)$	$z = \frac{\bar{x}_1-\bar{x}_2-D_0}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}$ If n_1 and n_2 are large, σ_1^2 and σ_2^2 may be replaced by s_1^2 and s_2^2	$\lvert z\rvert > z_{\alpha/2}$ $(z > z_\alpha)$ $(z < -z_\alpha)$	$\bar{x}_1 - \bar{x}_2 - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}} < \mu_1-\mu_2 < \bar{x}_1 -$ $\bar{x}_2 + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$ If n_1 and n_2 are large, σ_1^2 and σ_2^2 may be replaced by s_1^2 and s_2^2
6.	$N(\mu_1,\sigma_1^2)$ and $N(\mu_2,\sigma_2^2)$, where σ_1^2 and σ_2^2 are unknown, but we assume $\sigma_1^2 = \sigma_2^2$.	same as above	$t = \frac{\bar{x}_1-\bar{x}_2-D_0}{s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$ where $s_p^2 = \frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}$	$\lvert t\rvert > t_{\alpha/2,n_1+n_2-2}$ $(t > t_{\alpha}, n_1+n_2-2)$ $(t < -t_{\alpha}, n_1+n_2-2)$	$\bar{x}_1-\bar{x}_2-t_{\alpha/2,n_1+n_2-2}s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} < \mu_1-\mu_2 <$ $\bar{x}_1-\bar{x}_2+t_{\alpha/2,n_1+n_2-2}s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$
7.	Bin(p_1) and Bin(p_2) (large sample case-both $n_1, n_2 \geq 25$) $\hat{p}_1 = x_1/n_1, \hat{q}_1 = 1-\hat{p}_1$ $\hat{p}_2 = x_2/n_2, \hat{q}_2 = 1-\hat{p}_2$	$H_0: p_1-p_2 = D_0$ $H_1: p_1-p_2 \neq D_0$ $(p_1-p_2 > D_0)$ $(p_1-p_2 < D_0)$	$z = \begin{cases} \frac{\hat{p}_1-\hat{p}_2-D_0}{\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1}+\frac{\hat{p}_2\hat{q}_2}{n_2}}} & \text{if } D_0 \neq 0, \\ \frac{\hat{p}_1-\hat{p}_2}{\sqrt{\frac{\hat{p}\hat{q}}{n_1}+\frac{\hat{p}\hat{q}}{n_2}}} & \text{if } D_0 = 0, \end{cases}$ where $\hat{p} = \frac{x_1+x_2}{n_1+n_2}$	$\lvert z\rvert > z_{\alpha/2}$ $(z > z_\alpha)$ $(z < -z_\alpha)$	$\hat{p}_1-\hat{p}_2 - z_{\alpha/2}\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1}+\frac{\hat{p}_2\hat{q}_2}{n_2}} < p_1-p_2 <$ $\hat{p}_1-\hat{p}_2 + z_{\alpha/2}\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1}+\frac{\hat{p}_2\hat{q}_2}{n_2}}$
8.	$N(\mu_1,\sigma_1^2)$ and $N(\mu_2,\sigma_2^2)$, where μ_1 and μ_2 are unknown.	$H_0: \sigma_1^2 = \sigma_2^2$ $H_1: \sigma_1^2 \neq \sigma_2^2$ $(\sigma_1^2 > \sigma_2^2)$ $(\sigma_1^2 < \sigma_2^2)$	$F = \frac{s_1^2}{s_2^2}$ Take pop'n 1 to be that with the larger sample variance.	$F \geq F_{\alpha/2,n_1-1,n_2-1}$ $(F > F_{\alpha,n_1-1,n_2-1})$ do not reject	$\frac{s_1^2}{s_2^2}\frac{1}{F_{\alpha/2,n_1-1,n_2-1}} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{s_1^2}{s_2^2}F_{\alpha/2,n_2-1,n_1-1}$

All samples are assumed randomly drawn. In the two-population scenarios 5,6,7,and 8, the two samples are assumed independent of each other.

HW 5: Question 2

Problem: $X_1, \dots, X_n$ IID, (μ, σ^2) unknown, n large.

Find a $100(1-\alpha)\%$ CI for σ^2

Solution:

Recall that $\bar{X} \xrightarrow[n \to \infty]{D} \mathcal{N}(\mu, \frac{\sigma^2}{n})$ by CLT

↳ Ref: Rohatgi/Saleh textbook P. 301

So $\dfrac{\bar{X}}{\sigma/\sqrt{n}} \xrightarrow[n \to \infty]{D} \mathcal{N}(\mu, 1)$

So $0 \leq L < \sigma^2 < R$

$\Rightarrow 0 \leq \frac{1}{R} < \frac{1}{\sigma^2} < \frac{1}{L} \Rightarrow \frac{1}{\sqrt{R}} < \frac{1}{\sigma} < \frac{1}{\sqrt{L}}$

So $\underbrace{\dfrac{\bar{X}}{\sqrt{R}/\sqrt{n}}}_{\simeq \mu - z_{\frac{\alpha}{2}}} < \underbrace{\dfrac{\bar{X}}{\sigma/\sqrt{n}}}_{\sim \mathcal{N}(\mu, 1)} < \underbrace{\dfrac{\bar{X}}{\sqrt{L}/\sqrt{n}}}_{\simeq \mu + z_{\frac{\alpha}{2}}}$

use $\hat{\mu}$ instead of μ: $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$

So $\dfrac{\bar{X}}{\sqrt{R}/\sqrt{n}} = \bar{X} - z_{\frac{\alpha}{2}}$ and $\dfrac{\bar{X}}{\sqrt{L}/\sqrt{n}} = \bar{X} + z_{\frac{\alpha}{2}}$.

$$\Rightarrow \boxed{\begin{cases} L = \dfrac{n\,\bar{X}^2}{(\bar{X} + z_{\frac{\alpha}{2}})^2} \\[2ex] R = \dfrac{n\,\bar{X}^2}{(\bar{X} - z_{\frac{\alpha}{2}})^2} \end{cases}}$$