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15.470 Asset Pricing

Last Updated November 12, 2018

Fundamental Theory of Asset Pricing

Introduction

Definition (Positive Approach) Minimal restrictions of no arbitrage. Somewhat unique to finance due to wealth of data.
Definition (Normative Approach) Micro-founded model of portfolio choice ⇒ Decision rules ⇒ equilibrium restrictions on prices.

	Positive Approach (Reduced Form)	Normative Approach (Structure)
Theory is	useful for *relative* prices Derivative assets	useful for *absolute* prices Why is S&P returns at 7%/yr ?
Pros	Weak assumptions: learn from the data	economics more transparent, potentially more robust to policy changes
Cons	Prices are endogenous, world is always changing → conclusions may lack external validity Model may not apply in future	stronger assumptions → model will always be mis-specified

This Course: Demand for + Valuation of risky assets.
Ignores: frictions that make supply of risky assets hard (corpor. fin.)

- Financial frictions
- Capital structure: firm choice of debt/equity
- Moral hazard/separation of ownership and control (exec comp)
- Banking and liquidity

Definition (Gross Return) Change in value of a \$1 initial investment

$$R_{t+1} = \frac{\text{Payoff(t+1)}}{\text{Payoff(t)}} = \frac{X_{t+1}}{P_t}$$

- investor receives cash flow X_{t+1} without taking an action.
- can choose to liquidate position at market price X_{t+1}.
- can choose to reinvest cash flows at market price P_{t+1}.

Definition (Net Return) $r_t = 1 + R_t$.
Definition (Excess Return) $= r_t - r^f$.
Definition (Compound Return) $\Pi_{j=1}^k R_{t+j}$ = Return from t to $t+k$ where cash flows are reinvested.
Definition (Returns Stats)

Arithmetic: $\bar{R}_T^A = \frac{1}{T}\sum_{t=1}^T R_t$ (Artithm < Geom by Jensen)
Note: Arithmetic: quoted, not meaningful

Geometric: $\bar{R}_T^G = \left[\prod_{t=1}^T R_t\right]^{1/T} = \exp\left(\frac{1}{T}\sum_{t=1}^T \log R_t\right)$
Note: Geometric: captures compounding

Sample Variance: $\hat{\sigma^2} := \frac{1}{T-1}\sum_{t=1}^T \left[R_t - \frac{1}{T}\sum_{j=1}^T R_j\right]^2$

Stand. Dev/Volatility: $\hat{\sigma} = \sqrt{\hat{\sigma^2}}$

Skewness, Kurtosis: Higher moments.

Nonparametric: histograms, bootstrap, Monte-Carlo, resampling...

Definition (Risk Premium) $\pi_t = \mathbb{E}\left[r_t - r^f\right]$.

Definition (Sharpe Ratio) $SR_t = \frac{\mathbb{E}\left[r_t - r^f\right]}{\text{Std}(r_t - r^f)} = \frac{\mathbb{E}[\text{return}]}{\text{Unit Risk}}$.

Properties (Log Plot) $\log R(t+k) - \log R(t) = \sum_{i=t+1}^k \log R_i = \log R(t+1, t+k)$ cumulative log return over holding period $[t, t+k]$.

Vector Notation: $a = [a_1, \dots, a_n]' \in \mathbb{R}^n$
a ≥ 0: $a_i \geq 0\ \forall i$ ($\mathbf{a} \in \mathbb{R}_+^n$)
a > 0: $a_i \geq 0\ \forall i$ AND $a_i > 0$ for at least one i.
a ≫ 0: $a_i > 0\ \forall i$ ($\mathbf{a} \in \text{Int}\,\mathbb{R}_+^n$)

Arrow-Debreu State-Space Framework

Environment: **(M States)** $m = 1, \dots, M$

Two Dates: $t = 0, 1$

t = 1 State Space: $\Omega = \{\omega_1, \dots, \omega_M\} \implies$ **M states**

Prob. Measure: $\mathbb{P}$ over Ω, $p_m := \mathbb{P}(\omega_m)$, $\sum_{m=1}^M p_m = 1$.

Agents: **(K Agents)** $k = 1, 2, \dots, K$

Resources:

- **Information:** For now, assume **all** prior info in **same** $\mathbb{P}$. Assume **Rational** $\mathbb{E}$ & **Homog. Beliefs**
- **Endowment:** ∃ one perishable good in economy.
  $\mathbf{e^k} := [e_0^k, \mathbf{e_1^k}']' = [e_0^k, e_{11}^k, \dots, e_{1M}^k]' \in \mathbb{R}^{1+M}$
  e_0^k at $t = 0$ $\quad e_{1\omega}^k$ at $t = 1$ & state ω.
  Standard Portfolio Problem: $e_0^k > 0$, $\mathbf{e_1^k} = 0$.
  Nonnegative Endowment: $\mathbf{e^k} \in \mathbb{R}_+^{1+M}$ ($\mathbf{e^k} \geq 0$).
- **Production Technology:** Pay \$$I$ now, get $f_\omega(I)$ at $t = 1$ & state $\omega \implies y_0(I) = -I$; $\quad y_1(I) = [f_1(I), \dots, f_M(I)]'$
  Assume: $f_\omega(0) = 0$ (invest nothing get nothing)
  & $f'_\omega(I) \geq 0$, $f''_\omega(I) \leq 0$ (Increasing but Diminishing returns)

Choices: Consumption + Resource Alloc.

- **Consumption:** $\mathbf{c^k} := [c_0^k, \mathbf{c_1^k}']' = [c_0^k, c_{11}^k, \dots, c_{1M}^k]' \in \mathbb{R}^{1+M}$
  $c_0^k = e_0 - P'\theta$ $(t = 0)$ (How much I eat today)
  $c_{1\omega}^k = e_1 + D_1\theta|_\omega$ ($t = 1$, state ω) (How much I eat tomorrow)
- **Consumption Plan:** $\mathbf{c^k} := [c_0^k, \mathbf{c_1^k}']$
- **Consumption Path:** $[c_0^k, c_{1\omega}^k] \implies$ **What we Observe!**
- **Consump. Set:** $C := \{\mathbf{c^k} : \text{feas.}\} \subseteq \mathbb{R}^{1+M}$; Usual: $C = \mathbb{R}_+^{1+M}$
  Prop: C is a **closed** + **convex** subset of $\mathbb{R}^{1+M}$.
- **Budget Set:** Set of consumption plans given by purchasing θ:
  $B(e, \{D_1, P\}) := \{c \geq 0 : \theta \in \mathbb{R}^N, c_0^k = e_0 - P'\theta, c_1^k = e_1 + D_1\theta\}$

Preferences: How they make choices.
Rational Preference: binary relat $\succsim^k$ over consump $C = \mathbb{R}_+^{1+M}$ s.t.

- **Complete:** $a, b \in C \implies a \succsim^k b$ or $b \succsim^k a$ or both.
- **Reflexive:** $a \in C \implies a \succsim^k a$
- **Transitive:** $a \succsim^k b$ & $b \succsim^k c \implies a \succsim^k c$

Continuous Pref $\succsim$: ⇔ $\forall \{a_n\} \to a, \{b_n\} \to b \in C : a_n \succsim b_n \Rightarrow a \succsim b$
Utility Function: $u_k : C \to \mathbb{R}$, s.t.
$a, b \in C : a \succsim b \iff u_k(a) \geq u_k(b)$.
Strictly Monotonic: $u_k(c) > u_k(c')$, $\forall c > c'$.
Theorem (Debreu) $X \subseteq R^n : \succsim$ **rational** + **cont**
$\implies \succsim$ can be represented by a **continuous** utility function.

Securities Market: **(N Assets)** $n = 1, \dots, N$
Security: Financial Claim yielding payoff/**dividend** D_{1n} at $t = 1$.
Payoff Vector: $D_{1n} = [D_{11}, \dots, D_{1M}]' \in \mathbb{R}^M$, $D_{1\omega}$ at $t = 1$ state ω.
Market Structure: $D_1 = [D_{11}, \dots, D_{1N}] = [D_{1\omega n}]_{M \times N} \in \mathbb{R}^{M \times N}$
Price: $P_n \in \mathbb{R}$: price of security n at $t = 0$.
Price Vector: $P = [P_1, \dots, P_N]' \in \mathbb{R}^N$ at $t = 0$.
Portfolio: $\theta = [\theta_1, \dots, \theta_N] \in \mathbb{R}^N \Rightarrow \text{cost}_{t=0} = -P'\theta$, $\text{payoff}_{t=1} = D_1\theta$.
Short Sale: $\theta_j < 0$: borrow $|\theta_j|$ of asset $j (t = 0)$, pay $\theta_j D_{1\omega j} (t = 1)$.
B Matrix: $B := [-P', D]' \in \mathbb{R}^{(M+1) \times N}$.

Proposition $\boxed{c = e + B'\theta} = [e_0 - P'\theta, [e_{1m} + \sum_{n=1}^N D_{n1m}\theta_n]'_{M \times 1}]$

Frictionless Market:

- No access + transactions costs + taxes
- No position constraints + market impact + divisible goods
- No information asymmetry

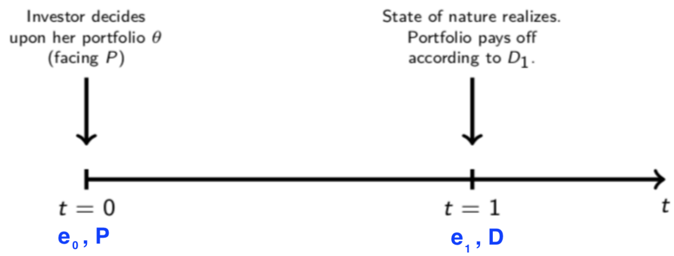
Market Equilibrium:

Optimization: $\max_\theta u_k(c^k)$ s.t. $c^k \in B(e, \{D_1, P\})$: Sol° $= \theta^k(P, e)$.
ex: $M = N = 1$, $D_1 = 1$, $P = 1/(1+r) \to$ borrow/lend at rate r.
ex: $M = N = 2$, $e_0 > 0$, $e_1 = 0$, c_0 fixed, wealth $w_0 = \text{Diag}(e_0 - c_0)$
$\to$ choose $c_{1\omega} = w_0 D_1 \text{Diag}(P^-1)\,\theta = w_0 R_1 \theta$ s.t. $\theta'\mathbf{1}_N = 1$.

Market Equilibrium: Supply = Demand
Market Clearing: $\sum_{k=1}^K \theta^k(P, e^k) = 0$ i.e. $\sum_{k=1}^K c^k = \sum_{k=1}^K e^k$.
$\implies$ Gives equilibrium prices $P(D_1, \mathbb{P}, \{u^k, e^k\}_{k=1}^K)$.

Pareto Dominance: Allocation c^k Pareto Dominates $c^{k'}$
$\iff u^k(c^k) \geq u^k(c^{k'}) \forall k$ and strict for one k.

Pareto Optimality: Allocation c^k is Pareto Optimal
$\iff c^k$ feas. $(\sum_k c^k = \sum_k e^k)$ & ∄ a feasible P.Dominating alloc.



Arbitrage

Replication:
Exclude Asset n: $\theta_{\setminus n} = [\theta_1, \dots, \theta_N]' \in \mathbb{R}^{N-1}$ portfolio excluding θ_n,
$D_{\setminus n} = [D'_1, \dots, D'_N]' \in \mathbb{R}^{M \times (N-1)}$ payoff matrix excluding D_n.
Definition (Redundant Security) Security n is redundant
$\iff \exists \theta_{\setminus n}$ s.t. $D_{\setminus n}\theta_{\setminus n} = D_n$

Definition (Our Setup) $\text{rank}(D_1) = N \leq M$:
$\implies$ drop redundant security (but possibly incomplete market).

Definition (Payoff Space C_1)
$C_1(D_1) := \{c_1 = D_1\theta \in \mathbb{R}^M : \theta \in \mathbb{R}^N\} = \text{span}(D_1, \dots, D_N) \subseteq \mathbb{R}^M$
Prop: $\dim C_1 = N$

Definition (Payoff Replicat°) Payoff c_1 = replicated/financed by θ
$\iff c_1 \in C_1(D_1) \iff \exists$ portfolio $\theta \in \mathbb{R}^N$ s.t. $c_1 = D_1\theta$

Definition (Complete Market) A securities market is complete
$\iff \forall$ payoff $c_1 \in \mathbb{R}^M, \exists \theta \in \mathbb{R}^N$ s.t. $D\theta = c_1$.
$\iff \text{span}(D_1, \dots, D_N) = \mathbb{R}^M$
$\iff \text{rank}(D) = M$ (i.e., need $N = M$)

Definition (State-Contingent Claims/Arrow-Debreu Securities)
State-ω contingent claim $e_\omega \in \mathbb{R}^M$ has payoff 1 in state ω, 0 otherwise.

Definition (Arrow-Debreu Market/Economy)
A securities market with a **complete set** of A-D securities:
$D^{AD} = \mathbb{I}_{N \times N} = \mathbb{I}_{M \times M}$ **(as N = M)**
Prop: An AD Market is complete.

Definition (State Price Vector ϕ)
ϕ_ω = price of e_ω at $t=0$.
Vector: $\phi = [\phi_1, \dots, \phi_M]' \in \mathbb{R}^M$.
Set of State Prices: $\Phi = \left\{\phi \in \mathbb{R}^M : \text{consistent with NA}\right\}$ $(P' = \Phi' D)$
Prop: Payoff $c = [c_1, \dots, c_M]' = \sum_{\omega=1}^M e_\omega c_\omega$
Prop: Price of Portfolio $\theta = [\theta_1, \dots, \theta_{M=N}]'$: $P = \sum_{n=1}^N \phi_n D_n$ (?)

Definition (Arbitrage)
Given market with structure D, price vector P:
Arbitrage = trading strategy θ at $t=0$ s.t.

1. Require no cash inflow at $t=0$: $P'\theta \le 0$
2. Generates positive cash flow at $t=1$: $D\theta \ge 0$

AND one of the ineqs is **strict**.
Note: Arbitrage = rely on prices+payoffs NOT probabilities
Arbitrage = scalable (frictionless market) + available to everyone

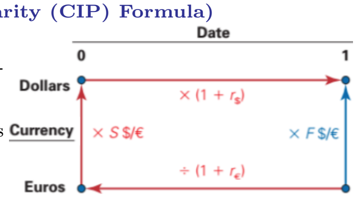
Principles (No Arbitrage) Frictionless market $\implies \nexists$ arbitrage

Proposition (Arbitrage Existence) $\iff \exists \theta \in \mathbb{R}^N$ s.t. $B\theta > 0$.
Type 1: $B\theta = [0, > 0]'$ **free** at $t=0$, **maybe** paid at $t=1$
Type 2: $B\theta = [>0, \ge 0]'$ **paid** at $t=0$, **maybe** paid at $t=1$

Proposition
Market Equilib + 1 agent w/ strictly monotonic pref $\implies \nexists$ arbitrage
$(\because)$ invest ∞ amount in it $\implies$ no equilibrium
Warning: $\nexists$ arbitrage $\not\Leftrightarrow$ Market Equilibrium

Example: (Covered Interest Parity (CIP) Formula)
- Borrow € today at int. rate $r_€$.
- Convert € to \$ at the current exchange rate S.
- Invest \$ at US interest rate $r_\$$.

$\implies$ Must have **SAME** price F as forward contract: $F = S \times \frac{1+r_\$}{1+r_€}$



Note: (Limits to Arbitrage)
Pure arbitrage $\exists$ only in **perfect** markets. In **practice**:
- Need capital + posting collateral.
- Shorting is costly! People are worried about default risk
- **Imperfect Information + Market Frictions** $\implies$ arbitrage strategies are capital intensive + risky:

- Arbitrageurs use leverage to invest more than their own \$.
- Can get margin calls before the arbitrage takes place.
- Short-selling is risky! No limited liability (∞ losses)

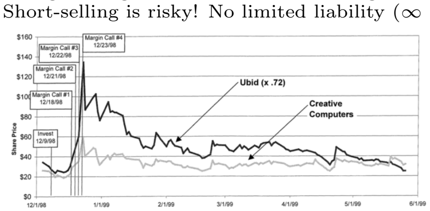


Figure 1. Paths of stock prices for Creative Computers and Ubid.

- "To avoid the costly margin calls, the arbitrageur would have had to post \$4.53 of excess cash for every \$1 of long position."
- Return with transaction costs/margin limits: 9.5%
- Return ignoring frictions: 45.9%

CIP Violations in the Data:
- Arbitrage is associated with opportunity costs
- Regulations $\implies$ only financial institutions can close the CIP arb.
- High capital requirements for trading to close CIP arb. (very high costs at quarter ends)

Fundamental Thm of Asset Pricing (FTAP)

Valuation Operator:
Definition (Asset Pricing Model) Mapping from a securitys payoff vector d to its price P: $P = V(d)$.
Definition (Pricing/valuation operator) $V: \mathbb{R}^M \supseteq C_1(D) \to \mathbb{R}^M$ $d \mapsto P = V(d)$.
Theorem Under No Arbitrage: $d, d_1, d_2 \in C_1(D), a, b \in \mathbb{R}$
Positivity $d > 0 \implies V(d) > 0$, and $d = 0 \implies V(d) = 0$
Law of One Price $d_1 = d_2 \implies V(d_1) = V(d_2)$
Monotonicity $d_1 \ge d_2 \implies V(d_1) \ge V(d_2)$
Linearity $V(a \cdot d_1 + b \cdot d_2) = aV(d_1) + bV(d_2)$
Theorem (FTAP) $\nexists$ arbitrage in market
$\iff \exists \phi \gg 0$ s.t. $P' = \phi' D$ (i.e., $P = D'\phi$).
$\phi \in \mathbb{R}^{M+1}$ = **State Price Vector** implied from D and P.
Note: Given ϕ, can price any traded asset (don't need $\mathbb{P}$ or $\mathbb{Q}$).
Proposition Redundant securities $\implies \phi$ is **not unique**.
May find ϕ that are not $\gg 0$. If $\nexists \phi \gg 0 \implies \exists$ arbitrage.
Definition (DCF/PV Formula) $P_n = \phi' D_n = \sum_\omega \phi_\omega D_{n\omega}$
Proposition $\phi \gg 0 \implies$ all AD prices are arbitrage-free.
Note: ϕ_ω = price of a hypothetical AD security (whose payoff may or may not be achievable).
If ϕ large in bad states $\implies$ insurance.
Theorem (Stiemke's Lemma) $\exists \phi \gg 0$ $(\in \mathbb{R}^m)$ s.t. $P' = \phi' D$
$\iff \nexists \theta \in \mathbb{R}^n$ s.t. $B\theta > 0$.
Example: (Incomplete Market) $D = (1,2,3)', P = 1$:
$\implies \Phi = \left\{\phi \gg 0 : \phi' D = 1\right\} = \left\{\phi \gg 0 : \phi_1 + 2\phi_2 + 3\phi_3 = 1\right\}$.
Price $D_2 = (2,2,2)' : P_2$
Example: (Incomplete Market II) Find price P_b of new security with payoff b s.t. $b \notin \text{span}(D_1)$ (i.e., not redundant):
$P_b = \left\{\phi b : \phi \gg 0, P' = \phi' D_1\right\}$ i.e., $\inf_{\phi \in \Phi \gg 0} \phi b < P_b < \sup_{\phi \in \Phi \gg 0} \phi b$
If Security b is redundant: $P_b = \{P\theta : \theta \in \mathbb{R}^n, D_1\theta = b\} = \phi b$.
Special Case – Complete Markets:
Proposition Complete market $\implies \phi = (D^{-1})'P$.
Theorem No Arbitrage $\implies \exists! \phi \in \mathbb{R}^M$ s.t. $P' = \phi' D$.
Furthermore, $\exists$ portfolio $\theta \in \mathbb{R}^N$ s.t. $\phi = D\theta$.
Note: If D has redundant columns: θ not unique BUT ϕ is unique
Proposition Complete Market + No Arbitrage $\implies \exists! \phi \gg 0$.

State-Price Density/Risk Neutral Measure

Definition (Risk-Free Asset) Payoff: $D_1 = \mathbf{1}_M \in \mathbb{R}^M$ i.e., $D_{1\omega} = 1$, $\forall \omega \in \Omega$, with price $P_1 = \frac{1}{1+r^f} = \sum_{\omega=1}^M \phi_\omega = \phi' \mathbf{1}_M$.
Definition (Risk-Neutral Measure) $\mathbb{Q} = \{q_\omega : \omega \in \Omega\}$, where $q_\omega = \frac{\phi_\omega}{\sum_{\omega'} \phi_{\omega'}} = (1+r^f)\phi_\omega$
Prop: $q_\omega > 0$, $\sum_\omega q_\omega = 1$ and $\mathbb{Q} \sim \mathbb{P}$ (agree on zero-measure sets).
Idea: Probabilities/riskiness hidden in state prices ϕ.
$\implies \mathbb{Q}$ = normalized state prices ($\neq \mathbb{P}$: tied to data/observed).
Risk-Neutral Pricing:

1. $P' = \phi' D \implies$ get state prices ϕ.
2. Get RF rate: $1 + r^f = \frac{1}{\sum_\omega \phi_\omega}$.
3. Construct $\mathbb{Q} = \left\{q_\omega = \frac{\phi_\omega}{\sum_{\omega'} \phi_{\omega'}} = (1+r^f)\phi_\omega\right\}$.
4. Price any asset with payoff vector D_n: $P_n = \frac{\mathbb{E}^{\mathbb{Q}}[D_n]}{1+r^f} = \frac{1}{1+r^f}\sum_\omega q_\omega D_{n\omega}$.
5. Get **Expected Return**: $1 + \bar{r}_n = (1+r^f)\frac{\mathbb{E}^{\mathbb{P}}[D_n]}{\mathbb{E}^{\mathbb{Q}}[D_n]}$.

Definition (State-Price Density/Stochastic Discount Factor η)
Idea: $P_n = \phi' D_n = \sum \phi_\omega D_{n\omega} = \sum p_\omega \frac{\phi_\omega}{p_\omega} D_{n\omega}$
$\implies \eta_\omega = \frac{\phi_\omega}{p_\omega}$, $\forall \omega \in \Omega$. **Prop:** $\eta_\omega > 0$, $\eta \in \mathbb{R}^M$, $\mathbb{E}^{\mathbb{P}}[\eta] = \frac{1}{1+r^f}$.
$\implies \eta$ = **PAIN INDEX!** η **small** $\to$ **good state;** η **big** $\to$ **bad** (how worried people are about the future)
Note: η is hard to observe (unless market is complete)
Proposition ($\mathbb{P} \sim \mathbb{Q}$) $\mathbb{E}^{\mathbb{P}}[X] = \frac{1}{1+r^f}\mathbb{E}^{\mathbb{Q}}\left[\frac{X}{\eta}\right]$, & $\phi_\omega = \frac{q_\omega}{1+r^f} = p_\omega \eta_\omega$
Note: $p_i > q_i \implies$ payoffs in state i are very valuable
Idea: η removes the probabilities from ϕ: it will add them with $\mathbb{E}^{\mathbb{P}}[\cdot]$.
SPD/SDF Pricing: you must know $\mathbb{P} = \{p_\omega\}$:

1. $P' = \phi' D \implies$ get state prices ϕ.
2. Get SPD/SDF: $\eta = \frac{\phi_\omega}{p_\omega}$.
3. Price any asset with payoff vector D_n: $P_n = \mathbb{E}^{\mathbb{P}}[\eta D_n] = \sum_\omega p_\omega \eta_\omega D_{n\omega}$.
4. Get **Expected Return**: $1 + \bar{r}_n = \frac{\mathbb{E}^{\mathbb{P}}[D_n]}{P_n} = \frac{\mathbb{E}^{\mathbb{P}}[D_n]}{\mathbb{E}^{\mathbb{P}}[\eta D_n]}$.
5. Get RF rate: $1 + r^f = \frac{1}{\mathbb{E}^{\mathbb{P}}[\eta]}$.

Theorem (Representation Thm) $\exists$ a positive pricing operator V
$\iff \exists$ risk-neutral measure $\mathbb{Q}$ & riskless asset r^f
$\iff \exists$ SPD/SDF $\eta \gg 0$.
Discounted Cash Flow (DCF)/Present Value (PV) Formula:
Definition (Discount Rate/Expected Rate of Return)
$1 + \bar{r}_n = \frac{\mathbb{E}^{\mathbb{P}}[D_n]}{P_n} = \frac{\mathbb{E}^{\mathbb{P}}[D_n]}{\mathbb{E}^{\mathbb{P}}[\eta D_n]} = (1+r^f)\frac{\mathbb{E}^{\mathbb{P}}[D_n]}{\mathbb{E}^{\mathbb{Q}}[D_n]} = \frac{\mathbb{E}^{\mathbb{Q}}[D_n/\eta]}{\mathbb{E}^{\mathbb{Q}}[D_n]}$.
Definition (Rate of Return) Random $\tilde{r}_n : r_{n\omega} = \frac{D_{n\omega}}{P_n} - 1$
Note: $\tilde{r}_{1\omega} = r^f$, $\mathbb{E}^{\mathbb{P}}[\tilde{r}_n] = \bar{r}_n$, and $1 + \tilde{r}_n = \frac{D_n}{P_n}$
Definition (Risk Premium) $\pi_n = \mathbb{E}^{\mathbb{P}}\left[\tilde{r}_n - r^f\right] = \bar{r}_n - r^f$.
Proposition (DCF/PV) $P_n = \frac{\mathbb{E}^{\mathbb{P}}[D_n]}{1+\bar{r}_n} = \frac{\sum_\omega p_\omega D_{n\omega}}{1+\bar{r}_n}$
$P_1 = \sum_\omega \phi_\omega = \frac{1}{1+r^f}$ and $\bar{r}_1 = \frac{\mathbb{E}^{\mathbb{P}}[D_1]}{P_1} - 1 = \frac{1}{P_1} - 1 = r^f_f$
Proposition (Risk Premium)
$\mathbb{E}^{\mathbb{P}}[1 + \tilde{r}_n] = (1+r^f)\left(1 - \text{Cov}^{\mathbb{P}}(\eta, 1+\tilde{r}_n)\right)$
$\implies \pi_n = \mathbb{E}^{\mathbb{P}}\left[\tilde{r}_n - r^f\right] = -(1+r^f)\text{Cov}^{\mathbb{P}}\left(\eta, \tilde{r}_n - r^f\right)$
and $\mathbb{E}^{\mathbb{Q}}[\tilde{r}_n] = r^f\ \forall n \longrightarrow \mathbb{E}^{\mathbb{Q}}\left[\tilde{r}_n - r^f\right] = 0\ \forall n$
$(\because)$ $P_n = \mathbb{E}^{\mathbb{P}}[\eta D_n] \implies 1 = \mathbb{E}^{\mathbb{P}}[\eta D_n / P_n] = \mathbb{E}^{\mathbb{P}}[\eta(1+\tilde{r}_n)]$
$1 = \mathbb{E}^{\mathbb{P}}[\eta]\,\mathbb{E}^{\mathbb{P}}[(1+\tilde{r}_n)] + \text{Cov}^{\mathbb{P}}(\eta, 1+\tilde{r}_n)$
Idea: Asset performing well in bad times earns lower returns.
"–" sign: if make money in bad state (insurance): pay for it!
Corollary (Irrelevance of Idiosyncratic Risk)
Decomposition: $D_n = \text{Proj}(D_n|\eta) + \varepsilon_n$ where $\varepsilon \perp \eta$ & $\mathbb{E}[\varepsilon] = 0$.
$P_j = P_k \iff \text{Proj}(D_j|\eta) = \text{Proj}(D_k|\eta)$
Note: $\rho(\eta, D_n) = 0 \implies \bar{r}_n = r^f$ (but $\tilde{r}_n \neq r^f$) and $P_n = \frac{\mathbb{E}^{\mathbb{P}}[D_n]}{1+r^f}$
Example: (Log-Normal Case) $\log \eta$ and $\log(1+\tilde{r}_n)$ jointly normal:
$\implies \mathbb{E}^{\mathbb{P}}[\log(1+\tilde{r}_n)] - \log(1+r^f) + \frac{1}{2}\text{Var}^{\mathbb{P}}(\log(1+\tilde{r}_n)) = -\text{Cov}^{\mathbb{P}}(\log \eta, \log(1+\tilde{r}_n))$
Theorem (Hansen-Jagannathan Bound) Sharpe Ratio of asset n:
$SR_n := \frac{\mathbb{E}^{\mathbb{P}}\left[\tilde{r}_n - r^f\right]}{\sqrt{\text{Var}^{\mathbb{P}}(\tilde{r}_n - r^f)}} \le \frac{\sqrt{\text{Var}^{\mathbb{P}}(\eta)}}{\mathbb{E}^{\mathbb{P}}[\eta]}$
$(\because)$ $\frac{\pi_n}{1+r^f} = -\text{Cov}^{\mathbb{P}}\left(\eta, \tilde{r}_n - r^f\right) \le -(-1) \cdot \sqrt{\text{Var}^{\mathbb{P}}(\eta)}\sqrt{\text{Var}^{\mathbb{P}}(\tilde{r}_n - r^f)}$
Definition (Entropy of a r.v.) $X > 0$:
$L^{\mathbb{P}}(X) = \log \mathbb{E}^{\mathbb{P}}[X] - \mathbb{E}^{\mathbb{P}}[\log X] \ge 0$.
Theorem (Entropy Bound) $L^{\mathbb{P}}(\eta) \ge \mathbb{E}^{\mathbb{P}}[\log(1+\tilde{r}_n)] - \log(1+r^f)$.
Note: Can observe $SR > 0.8 \implies \eta$ VERY volatile
Many models generate $\sigma(\eta)$ & $L^{\mathbb{P}}(\eta)$ much lower than bound

FTAP: Corporate Finance

Assumptions
(1) Not restricted to financial assets (allow agents to invest in real productive opportunities)
(2) Securities market: frictionless + complete
$\Longrightarrow$ complete set of AD securities traded with price vector $\phi \gg 0$
(3) Endowment $e = [e_0, e_1']'$.
(4) Firms = only defined by the production technologies they possess
y_0: investment into the production opportunity at $t=0$
$y_{1\omega} = y_\omega(y_0)$, $\forall \omega \in \Omega$: output from production at $t=1$, state ω
Assume: $y_\omega(0)=0$, $y_\omega'(\cdot)\ge 0$ and $y_\omega''(\cdot)<0$ **(diminishing returns)**
Production Vector: $y_1 = [y_{11}, \dots, y_{1M}]'$
(5) Agent wants to maximize utility $u(c)$

Definition (Investment NPV) $v = \phi' y - y_0 = \sum_\omega \phi_\omega y_\omega(y_0) - y_0$

Definition (Agent's $t=0$ Wealth)
$w = e_0 - y_0 + \phi'(e_1 + y_1) = e_0 + \phi' e_1 + v$

Definition (Agent's Optimization Pb) $\max_{y_0,c_0,c_1} u([c_0, c_1']')$ s.t. $w = c_0 + \phi' c_1$
Solution:
(1) Choose y_0 to maximize $t=0$ wealth (NPV of production)
FOC (dv/dy_0): $1 = \phi' y'(y_0) = \mathbb{E}^{\mathbb{P}}[\eta_\omega \cdot y_\omega'(y_0)] = \frac{1}{1+r^f}\mathbb{E}^{\mathbb{Q}}[y_\omega'(y_0)]$
$y(\cdot)$ concave $\Longrightarrow v(\cdot)$ concave $\Longrightarrow$ unique solution
⇒ Optimal prod° decision = indep of agent's consumption decisions! Only depends on prod function & state prices
(2) Choose c to maximize utility $u(c)$

Corporate Investment Decisions:
Production opportunities are owned by firms:
$j = 1,\dots,F$ firms with Prod Tech $y_j(y_{j0}) \in \mathbb{R}^M$.
s_{kj} = share of firm j owned by agent k: $\sum_k s_{kj} = 1, \forall j$
Firm's investment NPV at $t=0$: $v_j = \phi' y_j(y_{j0}) - y_{j0}$, $\forall k$
Agent k's wealth: $w_k = e_{k0} + \phi' e_{k1} + \sum_j s_{kj} v_j$
Firm's investment decision: if firm j owned only by agent k,
Decision on firm's investment = maximize v_j: $\phi' y_j(y_{j0}) \overset{!}{=} 1$
Decision is indep of agent's endowment & preferences
⇒ firm's optimal investment decision indep of who owns it.
Theorem (Maximize Current Market Value) Frictionless + Complete market: ∃ unanimity among firm's shareholders on investment decisions (maximize NPV) separate ownership/mngmt!

Financing Decisions (Capital Structure):
Definition (Financing Decision) How firm raises funds for investmt

Definition (Capital Structure) Mix of securities issued by firm

Assumptions Firm financed by debt & equities:
(1) $d_0 = D$ debt issued at $t=0$
(2) $e_0 = E$ shareholder's equity at $t=0$
So: $y_0 = d_0 + e_0$
(3) d_1 & $e_1 \in \mathbb{R}^M$ (debt & equity at $t=1$): $y_1 = d_1 + e_1$

Proposition $D = \phi' d_1$, $E = \phi' e_1$ and $V = D + E = \phi' y_1$.
NPV(Equity)= $\phi'(y_1 - d_1) - e_0 = \phi' y_1 - (d_0+e_0) = \phi' y_1 - y_0$

Theorem (Modigliani-Miller) Frictionless + Complete market: firm's NPV determined only by investment decisions (indep of cap structure)

Example: (Labor vs Wage) Firm: hire L labor at wage W
$\Longrightarrow$ produce output $Y(L) = AL^\alpha$ $(\alpha<1)$, $\log A \sim N(\bar A, \sigma_A^2)$
Can borrow at r^f, and Log-SDF: $\log M = \delta + \varepsilon$, $\varepsilon \sim N(0,\sigma_\varepsilon^2)$
Assume correlation between $\log A$ and ε: $\sigma_{A\varepsilon}$.
Firm's problem: $\max_L -WL + \mathbb{E}[MAL^\alpha] \Longrightarrow L = \left(\frac{\alpha \mathbb{E}[MA]}{W}\right)^{1/(1-\alpha)}$,
where: $\mathbb{E}[MA] = \exp\left(\delta + \bar A + \frac12\sigma_A^2 + \underbrace{\frac12\sigma_\varepsilon^2 + \sigma_{A\varepsilon}}_{\text{ignore if } \sigma_\varepsilon^2=0}\right)$

Math Tricks

Theorem (Iterated Expectations) X a r.v., $\mathcal{F}_1 \subset \underset{\text{more info}}{\mathcal{F}_2}$
$\mathbb{E}[X|\mathcal{F}_1] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_2]|\mathcal{F}_1]$
Definition (Normal Distribution) $X\sim N(\mu,\sigma^2)$:
PDF: $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$.
Transfo: $Y = aX+b \Longrightarrow Y\sim N(a\mu+b, a^2\sigma^2)$.
MGF: $M_X(t) = \mathbb{E}\left[e^{-tX}\right] = \exp\left(\mu t + \frac12\sigma^2t^2\right)$
Prop: $X\perp Y \Longrightarrow M_{X+Y}(t) = M_X(t)\cdot M_Y(t)$
Definition (Multivariate Normal) $X\sim N(\mu,\Sigma)$, $X,\mu\in\mathbb{R}^n$:
PDF: $f_X(x) = \frac{1}{(2\pi)^{n/2}\sqrt{\det(\Sigma)}}\exp\left(-\frac12(x-\mu)'\Sigma^{-1}(x-\mu)\right)$.
MGF: $M_X(t) = \mathbb{E}\left[e^{-\sum_{i=1}^n t_iX_i}\right] = \exp\left(t'\mu + \frac12 t'\Sigma t\right)$, with $t\in\mathbb{R}^n$.
Correlation: X, Y jointly normal with correl ρ:
$M_{X+Y}(t) = \exp\left((\mu_1+\mu_2)t + \frac12(\sigma_1^2+\sigma_2^2+2\rho)t^2\right)$
$\Longrightarrow (X+Y)\sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2+2\rho)$
Example: (CARA: Normal) $U = -\mathbb{E}[\exp(-aZ)]$, $Z\sim N(\mu,\sigma^2)$
$\Longrightarrow U = -M_Z(a) = -\exp\left(-a\mu + \frac12 a^2\sigma^2\right)$; $-\frac1a\log U = \mu - \frac12 a\sigma^2$
Example: (CRRA: Log-Norm) $U = -\mathbb{E}\left[\frac{Z^{1-\gamma}}{1-\gamma}\right]$, $\log Z\sim N(\mu,\sigma^2)$
$\Longrightarrow U = \frac{1}{1-\gamma}M_{(1-\gamma)Z}(1) = \frac{1}{1-\gamma}\exp\left((1-\gamma)\mu + \frac12(1-\gamma)^2\sigma^2\right)$

FTAP: Fundamental Value of a Stocks

Cash Flows: $X_t = D_t + P_t$ (dividends + share price)
Returns: $R_t = \frac{X_t}{P_t} = \frac{D_t+P_t}{P_t}$.
Assumptions No Arbitrage holds $\forall t \Longrightarrow \exists \eta_{t+k} \forall k$
Theorem (PV of Future Stock Payoffs)
$P_t = \mathbb{E}_t^{\mathbb{P}}[\eta_{t+1}(P_{t+1}+D_{t+1})]$
$\Longrightarrow P_t = \mathbb{E}_t^{\mathbb{P}}\left[\sum_{k=1}^\infty \eta_{t:t+k}D_{t+k}\right]$ with $\eta_{t:t+k} := \prod_{j=1}^k \eta_{t+j}$
& $P_t = \mathbb{E}_t^{\mathbb{Q}}\left[\sum_{k=1}^\infty \frac{D_{t+k}}{(1+r^f)^k}\right]$
$(\because)$ $P_t = \mathbb{E}_t\left[\sum_{k=1}^T(\eta_{t+1}\cdots\eta_{t+k})D_{t+k}\right] + \mathbb{E}_t[\eta_{t+1}\cdots\eta_{t+T}P_{t+T}]_{\to 0}$
Definition (k^{th} Period Return)
$\mathbb{E}_t^{\mathbb{P}}\left[R_{t:t+k}^{(k)}\right] = \frac{\mathbb{E}_t^{\mathbb{P}}[D_{t+k}]}{\mathbb{E}_t^{\mathbb{P}}[\eta_{t:t+k}D_{t+k}]} = \frac{\mathbb{E}_t[\text{Payoff}]}{\text{Price}}$
$\Longrightarrow P_t^{(k)} = \mathbb{E}_t^{\mathbb{P}}[\eta_{t:t+k}D_{t+k}] = \mathbb{E}_t^{\mathbb{P}}[D_{t+k}]/\mathbb{E}_t^{\mathbb{P}}\left[R_{t:t+k}^{(k)}\right]$
$\longrightarrow$ Price of 1 divident k periods in future
Theorem (DCF of Future Stock Payoffs)
$P_t = \mathbb{E}_t^{\mathbb{P}}\left[\sum_{k=1}^\infty\eta_{t:t+k}D_{t+k}\right] = \sum_{k=1}^\infty \mathbb{E}_t^{\mathbb{P}}[D_{t+k}]/\mathbb{E}_t^{\mathbb{P}}\left[R_{t:t+k}^{(k)}\right]$

FTAP: Fixed Income Securities

Idea: Assume default-free securities (e.g., US T-Bills)
Definition (Bond) Coupon payments + principal/par/face value
No coupon $\longrightarrow$ **Zero-Coupon/Pure-Discount Bond**
Zero-Coupon Bonds (ZCB):
$P_{N,t}$ = price of N-period bond at time t that pays $FV = 1\$$ at $t+N$.
Proposition Any N-period default-free coupon bond = portfolio of zero coupon bonds: $Price_t = \sum_{j=1}^N CF_{t+j}\cdot P_{j,t}$

Assumptions $\eta = \eta(s)$ depends on state variable at t
(1st order) Markov State Variables: $s_t\in\{1,\dots,S\}$.
$\pi(s_{t+1},s_t) := \mathbb{P}(s_{t+1}=s|s_t,\text{past}) = \mathbb{P}(s_{t+1}=s|s_t)$
$\Longrightarrow P_{N,t} = P_N(s_t)$ (price depends on maturity + state only)
No Arbitrage: Price = $\mathbb{E}$[discounted payoffs]
Proposition (1 period) Take $N=1$, state $s_t = j$:
$P_{1,t} = P_1(j) = \mathbb{E}_t[\eta_{t+1}]$
$(\because)$ $P_1(j) = \sum_{s=1}^S \mathbb{P}(s_{t+1}=s|s_t=j)\eta(s)\cdot(1\$) = \sum_{s=1}^S\pi(s,j)\eta(s)$
Proposition (N periods)
$P_{N,t} = \mathbb{E}_t[\eta_{t+1}P_{N-1,t+1}] = \mathbb{E}_t[\eta_{t+1}\times\cdots\times\eta_{t+N}] = \mathbb{E}_t[\eta_{t:t+N}]$
$(\because)$ $P_2(s_t) = \sum_{s=1}^S \pi(s_{t+1},s_t)P_1(s_{t+1}) = \mathbb{E}_t[\eta_{t+1}\mathbb{E}_{t+1}[\eta_{t+2}]]$

Yield Curve (YC)/ Term Structure of Interest Rates:

Definition (YTM) For ZCB, Yield To Maturity = per-period gross discount rate → geom. avg of cumul return (hold ZCB until maturity)
$Y_{N,t} = \left[\frac{1}{P_{N,t}}\right]^{1/N} = [\text{payoff/price}]^{1/N} \longrightarrow P_{N,t} = \left[\frac{1}{Y_{N,t}}\right]^N$
Prop: Prices & Yields/returns move in opposite directions

Definition (Log-Framework) $p_{N,t} = \log P_{N,t}$, $y_{N,t} = \log Y_{N,t}$
Prop: $y_{N,t} = -\frac1N p_{N,t} \iff p_{N,t} = -N\cdot y_{N,t}$
Definition (Elasticity) of the bond price w.r.t. the yield:
$\frac{dp_{N,t}}{dy_{N,t}} = -N \Rightarrow$ long maturity ZCB is more sensitive to same change in y

Properties
- YTM = avg rate of return over the life of the loan: YTM across maturities → different units.
- ZCB prices: $P_{N,t}$ = exchange rate between 1\$ today & 1\$ at $t+N$.
- $\frac{P_{2,t}}{P_{1,t}} = \frac{1\$_{t+2}}{1\$_{t+1}}$ low $\Longrightarrow$ cheap to transfer cash from $t+1$ to $t+2$

Ex: recession at t which will end by $t+2$
Proposition (YC Recipe) $P_{N,t}=\mathbb{E}_t[\eta_{t:t+N}]$: YC⇔moments of SDF
(1) Define State Variables: x_t (data ⇒ need at least 3).
(2) Assume SDF $\eta_t = \eta(x_t)$ or Log SDF $m_t = \log\eta_t$.
(3) Give law of motion for x_t under $\mathbb{P}$ (use $\pi(x_{t+1},x_t)$)
OR Give law of motion for x_t under $\mathbb{Q}$ (use r^f)
(4) Sol°: Iterate on pricing eq. $P_N(x_t) = \mathbb{E}_t[\eta(x_{t+1})P_{N-1}(x_{t+1})|x_t]$
(5) Guess $p_{N,t} = \log(P_{N,t}) \sim$ affine in x_t & find coeffs

Example: (Vasicek model, 1977)
(1) **One State Variable:** x_t
(2) **Assume Log SDF:** $m_{t+1} = \log(\eta_{t+1}) = -x_t - \frac12\left(\frac\lambda\sigma\right)^2 - \frac\lambda\sigma\varepsilon_{t+1}$.
(3) **AR(1)** under $\mathbb{P}$: $x_{t+1} = \mu + \phi x_t + \sigma\varepsilon_{t+1}$; with $\phi<1$, $\varepsilon_t \overset{\text{iid}}{\sim} N(0,1)$

(4) **Sol°:** Iterate on $p_{N,t} = \log\mathbb{E}_t[\exp(m_{t+1}+p_{N-1,t+1})|x_t]$
Use: $(x_{t+1}|\mathcal{F}_t)\sim N(\mu+\phi x_t,\sigma)$ and $(m_{t+1}|\mathcal{F}_t)\sim N\left(-x - \frac12\left(\frac\lambda\sigma\right)^2, \frac\lambda\sigma\right)$
$p_{1,t} = \log\mathbb{E}_t[\exp(m_{t+1})] = -x_t - \frac12\left(\frac\lambda\sigma\right)^2 + \frac12\left(\frac\lambda\sigma\right)^2 = 0 - 1\cdot x_t$
$\Longrightarrow$ **Short Rate:** $x_t = y_{1t} = \log(1+r^f) \longrightarrow$ **mean-reverting** AR(1)
$p_{2,t} = \log\mathbb{E}_t[\exp(m_{t+1}+p_{1,t+1})] = \log\mathbb{E}_t[\exp(m_{t+1}-x_{t+1})]$
$= -(1+\phi)x_t + \left[-\frac12\left(\frac\lambda\sigma\right)^2 + \mu + \frac12\left(\frac\lambda\sigma+\sigma\right)^2\right] = A_2 + B_2\cdot x_t$

(5) **Guess** $p_{n,t} = A_n + B_n\cdot x_t$:
$p_{n,t} = A_n + B_n\cdot x_t \Longrightarrow p_{n+1,t} = A_{n+1} + B_{n+1}\cdot x_t$ with:
$B_n = -1 + \phi B_{n-1} = -\frac{1-\phi^n}{1-\phi}$
$A_n = A_{n-1} + B_{n-1}(\mu-\lambda) + \frac12 B_{n-1}^2\sigma^2$

Example: (Cox-Ingersoll-Ross, 1985)
(1) **One State Variable:** x_t
(2) **Assume Log SDF:** $m_{t+1} = -x_t - \frac12\left(\frac\lambda\sigma\right)^2 x_t - \left(\frac\lambda\sigma\right)x_t^{0.5}\varepsilon_{t+1}$.
(3) **Under $\mathbb{P}$:** $x_{t+1} = \mu + \phi x_t + \sigma x_t^{0.5}\varepsilon_{t+1}$; with $\phi<1$, $\varepsilon_t \overset{\text{iid}}{\sim} N(0,1)$

(4) **Sol°:** Iterate on $p_{N,t} = \log\mathbb{E}_t[\exp(m_{t+1}+p_{N-1,t+1})|x_t]$
Use: $(m_{t+1}|x_t) = cst - \left(\frac\lambda\sigma\right)x_t^{0.5}\varepsilon_{t+1}$ and $(p_{N-1,t+1}|x_t)\perp\varepsilon_{t+1}$
$\Rightarrow (m_{t+1}+p_{N-1,t+1}|x_t)\sim N(\mathbb{E}_t[m_{t+1}+p_{N-1,t+1}], \text{SD}_t[m_{t+1}+p_{N-1,t+1}])$
$\Longrightarrow p_{N,t} = \mathbb{E}_t[m_{t+1}+p_{N-1,t+1}] + \frac12\text{Var}_t[m_{t+1}+p_{N-1,t+1}]$
$\Longrightarrow p_{N,t} = \mathbb{E}_t[m_{t+1}+p_{N-1,t+1}] + \frac12\text{Var}_t[m_{t+1}] + \frac12\text{Var}_t[p_{N-1,t+1}] + \text{Cov}_t(m_{t+1},p_{N-1,t+1})$
$\Longrightarrow p_{1,t} = \mathbb{E}_t[m_{t+1}] + \frac12\text{Var}_t[m_{t+1}]$ therefore:
$p_{N,t} = p_{1,t} + \mathbb{E}_t[p_{N-1,t+1}] + \frac12\text{Var}_t[p_{N-1,t+1}] + \text{Cov}_t(m_{t+1},p_{N-1,t+1})$
$\Longrightarrow p_{1,t} = -x_t - \frac12\left(\frac\lambda\sigma\right)^2x_t + \frac12\left(\frac\lambda\sigma\right)^2x_t = -x_t$
$\Longrightarrow$ **Short Rate:** $x_t = y_{1t} = \log(1+r^f) \longrightarrow$ **mean-reverting** AR(1)

(5) **Guess** $p_{n,t} = A_n + B_n\cdot x_t$, therefore:
$p_{n+1,t} = p_{1,t} + \mathbb{E}_t[p_{n,t+1}] + \frac12\text{Var}_t[p_{n,t+1}] + \text{Cov}_t(m_{t+1},p_{n,t+1})$
$= -x_t + [A_n + B_n(\mu+\phi x_t)] + \frac12[B_n^2\sigma^2x_t] + [-B_n\lambda x_t]$
$= \underbrace{[A_n+\mu B_n]}_{A_{n+1}} + \underbrace{[-1+(\phi-\lambda)B_n + (1/2)\sigma^2B_n^2]}_{B_{n+1}}x_t$

Proposition (Bond Pricing with Real Returns)
Real: SDF=η_t^r, Price=P_t^r $\longrightarrow$ $P_{n,t}^r = \mathbb{E}_t\left[\eta_{t+1}^r P_{n-1,t+1}^r\right]$
Nominal: SDF=η_t, Price=P_t and **Price Level:** Π_t
$\implies P_{n,t} = P_{n,t}^r \Pi_t = \mathbb{E}_t\left[\eta_{t+1}^r P_{n-1,t+1}\frac{\Pi_t}{\Pi_{t+1}}\right]$
$\eta_{t+1} = \eta_{t+1}^r \frac{\Pi_t}{\Pi_{t+1}} =: \eta_{t+1}^r \frac{1}{1+\pi_{t+1}}$

Proposition (Fisher Eq.) If inflation risk uncorrelated with risk:
$\frac{1}{1+i_t}(1+\mathbb{E}_t[\pi_{t+1}]) = \frac{1}{1+r_t} \quad \implies \quad r_t \approx i_t - \mathbb{E}_t[\pi_{t+1}]$
($\because$) Take expectation above: $\mathbb{E}_t\left[\eta_{t+1}(1+\pi_{t+1})\right] = \mathbb{E}_t\left[\eta_{t+1}^r \cdot 1\right]$

FTAP: Options

Definition (Derivative Security) Contract whose value derives from the price of another security or observable outcome.

Definition (Underlying Asset) X_t = Payoff of asset at time $T > t$, s.t. $f(X_T)$ = payoff of derivative security at $T > t$, $f(\cdot)$ known.
Note: Payoff at $T > t$ can be path dependent: $f(X_{t+1}, \dots, X_T)$

Proposition (Derivative Price) Given η: $P_t^D = \mathbb{E}_t\left[\eta_{t:T} \cdot f(X_T)\right]$
Note: Often, derivative = redund asset $\Rightarrow$ use replicating portfolio

Definition (Long Forward Contract) Obligation to buy an underlying asset at a pre-specified price K at time T.
Prop: Payoff = $S_T - K$

Definition (Credit Default Swaps) Insure debt-holder against losses from default.

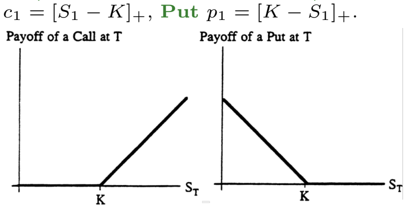
Definition (Interest Rate Swaps) Insure investor against interest rate risk: exchange (fixed set of cash payments)
$\longleftrightarrow$ (floating payments tied to interest rates)

Proposition (Arrow-Debreu) • AD security = derivative security
($\because$) Underlying Asset = is the state of nature $\longrightarrow$ see HW2-Q3b
• ANY Derivative = Portfolio of AD securit: $P^D = \sum_{\omega=1}^M \phi_\omega \cdot f(X_\omega)$

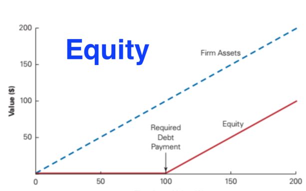
Options:

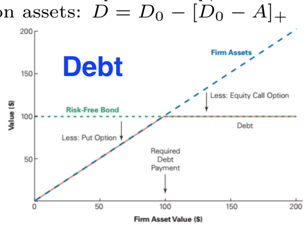
Definition (Stock/Underlying Asset)
S_0 = price at $t=0$ $\qquad$ S_1 (or S) = payoff at $t=1$.

Definition (European Call/Put Option) On the stock: Contract giving buyer the right to buy/sell stock from/to seller of option at $t = T$ & price K.
$T = 1$: **maturity/exercise date** ; K = **strike/exercise price**
Payoff: **Call** $c_1 = [S_1 - K]_+$, **Put** $p_1 = [K - S_1]_+$.



Example: Payoff Depends on Price of Underlying Asset at $t = 1$
Straddle (V) = call(K) + put(K): $[S-K]_+ + [K-S]_+$
Butterfly (Λ) = call($K-\delta$) - 2 calls(K) + put($K+\delta$):
$[S-K+\delta]_+ - 2[S-K]_+ + [K-S-\delta]_+$
$\star\star\star$ Let D_0 = required debt payment on firm:
Equity = **Call** option on assets of firm: $E = [A - D_0, 0]_+$
Debt = **RF Bond** − **put** option on assets: $D = D_0 - [D_0 - A]_+$





Definition (Intrinsic Value) Call: $I = S - K$; Put: $I = K - S$
In-The-Money: $I > 0$, $S > K$ (call), $K > S$ (put)
At-The-Money: $I = 0$, $S = K$ (call), $K = S$ (put)
Out-of-The-Money: $I > 0$, $S < K$ (call), $K < S$ (put)

Pricing Properties:

Proposition (Arbitrage Pricing Properties of Options)
$c(S,K) = V(c_1)$: **call price** ; $p(S,K) = V(p_1)$: **put price**
• Option prices are ≥ 0: $c(S,K) \geq 0$ and $p(S,K) \geq 0$.
• $c(S,K) \searrow$ in K and $p(S,K) \nearrow$ in K.
($\because$)
$\forall K_1 > K_2 \implies c(S,K_1) = V([S-K_1]_+) \leq V([S-K_2]_+) = c(S,K_2)$
• $c(S,K)$ and $p(S,K)$ are convex in K.
($\because$) $c(S,K) = \sum_\omega \phi_\omega (S_\omega - K)_+$
$\implies c(S, \lambda K + (1-\lambda)K') = \sum_\omega \phi_\omega (S_\omega - \lambda K - (1-\lambda)K')_+$
$\leq \sum_\omega \phi_\omega \left[\lambda(S_\omega - K)_+ + (1-\lambda)(S_\omega - K')_+\right]$

Proposition (Portfolio of Options) Let $\theta > 0$: portf of N assets; Price $S = [S_1, \dots, S_N]' > 0$; Strike $K = [K_1, \dots, K_N] > 0$. Then, $c(S'\theta, K'\theta) \leq \sum_{i=1}^N \theta_i c(S_i, K_i)$ and $p(S'\theta, K'\theta) \leq \sum_{i=1}^N \theta_i p(S_i, K_i)$
Note: Option on a portfolio $\leq$ Portf of options on assets in portfolio
($\because$) Payoff of option on portfolio $= [(S-K)'\theta]_+ = [\sum_i (S_i - K_i)\theta_i]_+$
$\leq \sum_i [S_i - K_i]_+ \theta_i$ = payoff of portfolio of options on each assets

Proposition (Option Price Bounds) $S \geq c(S,K)$
If $\exists$ riskless bond r^f: $\left[S - \frac{K}{1+r^f}\right]_+ \leq c(S,K) \leq S$
($\because$) Long 1 stock; Short K bonds.
Payoff $= S - K$ ($t=1$) ; Price $= S - K/(1+r^f)$ ($t=0$)
$c_1 = [S-K]_+ \geq S - K \implies c(S,K) = V(c_1) \geq S - K/(1+r^f)$

Proposition (Put-Call Parity) If $\exists$ riskless bond r^f:
No Dividend: $c(S,K) + \frac{K}{1+r^f} = p(S,K) + S$
With Dividend: D at $t=0$: $c(S,K) + \frac{K}{1+r^f} + D = p(S,K) + S$
($\because$) • Long 1 Call(K) + K Bonds
• Long 1 Put(K) + 1 Stock
$\implies$ **SAME** Payoff at $t=1$: K (if $S \leq K$) and S (if $S > K$)

Early Exercise:

Definition (American Option) Buyer can exercise at any $t \leq T$.
Price: $C(S,K)$ American Call , $P(S,K)$ American Put
Assume: $t_0 = T - 1$: can exercise now or wait $\leftrightarrow$ European option
Prop: $C(S,K) \geq c(S,K)$; $P(S,K) \geq p(S,K)$ ('>' if $\mathbb{P}$(early exerc)> 0)

Definition (Dividend) Payoff prior from the stock (before maturity)
$\implies$ Dividends can influence early exercise & value of American opts

Proposition (No Dividend: Call) $r^f > 0 \Rightarrow$ Do **NOT** exercise early
($\because$) $c_1^{early} = S - K \leq S - \frac{K}{1+r^f} \leq \left[S - \frac{K}{1+r^f}\right]_+ = c_1^{eur}$
1st Ineq: Pay strike price now, not later. Last Ineq: Give up the opt not to exerc at maturity $\implies V(c_1^{early}) \leq V(c_1^{eur})$

Proposition (No Dividend: Put) $r^f > 0 \Rightarrow$ **CAN** exercise early
($\because$) $P(S,K)$=max$\{K$-$S, p(S,K)\}$=max$\left\{K - S, \frac{K}{1+r^f} - S + c(S,K)\right\}$
Optimal if: $\frac{r^f}{1+r^f}K \geq c(S,K)$. Ex: K much bigger than S
Gain: Get strike price now, not later.
Loss: Give up the opt not to exercise at maturity

Proposition (With Dividend:) D= Divid ($t=0$), S=ex-divid price
Call: $C(S,D,K) = \max\{S + D - K, c(S,K)\}$
Put: $P(S,D,K) = \max\{K - S - D, p(S,K)\}$
($\because$) American Call: 2 choices at $t=0$
1) Exercise & get: dividend $D + S$ (sell stock ex-dividend)
2) Hold option to maturity ($t=1$).
$\implies$ Divids induce early exerc for calls & delay early exerc for puts

Complete Markets:
Recall: Complete Market $\implies$ $\exists$ a unique state price vector ϕ.
Note: If $\exists$ RF bond, then market is complete.

Proposition (European Call Price) $c(S,K) = \frac{\mathbb{E}^{\mathbb{Q}}[[S-K]_+]}{1+r^f}$
($\because$) $c(S,K) = \sum_\omega \phi_\omega (S_\omega - K)_+$, where S_ω=stock price, $t=1$, state ω

Theorem (Binomial Pricing)
Assume: $\exists$ RF bond w/: $t=1$ payoff 1, $t=0$ price $B = \frac{1}{1+r^f}$
Stock price: binomial process $S_1 = uS$ (w.p.p) and $S_1 = dS$ (w.p.$1-p$)
Note: u&d=gross return on stock: NA $\implies d < 1 + r^f < u$
$\implies c(S,K) = \phi_u[uS-K]_+ + \phi_d[dS-K]_+$ with
$\begin{cases} c_u = [uS-K]_+ \\ c_d = [dS-K]_+ \end{cases}$ $\begin{cases} \phi_u = \frac{1}{1+r^f}\frac{1+r^f-d}{u-d} \\ \phi_d = \frac{1}{1+r^f}\frac{u-1-r^f}{u-d} \end{cases}$ ($\because$) $\begin{cases} S = \phi_u uS + \phi_d dS \\ \frac{1}{1+r^f} = \phi_u + \phi_d \end{cases}$

Corollary (Replication Proof) Portfolio $\theta = [\theta_S, \theta_B]'$
$\implies c(S,K) = \theta_S S + \theta_B \frac{1}{1+r^f} = \frac{1}{1+r^f}\left(\frac{1+r^f-d}{u-d}c_u + \frac{u-1-r^f}{u-d}c_d\right)$
with $\begin{cases} \text{payoff}_u = \theta_S uS + \theta_B \overset{!}{=} c_u \\ \text{payoff}_d = \theta_S dS + \theta_B \overset{!}{=} c_d \end{cases}$ ($\because$) $\begin{cases} \theta_S = \frac{c_u - c_d}{(u-d)S} \\ \theta_B = \frac{uc_d - dc_u}{u-d} \end{cases}$

Corollary (Risk-Neutral Proof) Given State Prices ϕ:
$c(S,K) = \frac{\mathbb{E}^{\mathbb{Q}}[[S-K]_+]}{1+r^f} = \frac{qc_u + (1-q)c_d}{1+r^f}$ ($\because$) $q = \frac{\phi_u}{\phi_u + \phi_d} = \frac{1+r^f-d}{u-d}$

Market Structure - Completing Markets with Options:

Definition (State-Index Security - SIS) Security/Portfolio with state-separating payoff X: $\quad X_\omega = X_{\omega'} \iff \omega = \omega'$
Assume: WLOG $X_\omega < X_{\omega'}$ $\forall \omega < \omega'$

Example: (European Option on SIS) Eur. Call Option on SIS X:
Strike Price: $K = X_\omega \implies$ **Payoff:** (nonzero for states $\geq \omega + 1$)
$c_1 = [X - X_\omega]_+ = [0, \dots, 0, X_{\omega+1} - X_\omega, \dots, X_M - X_\omega]' \in \mathbb{R}^M$

Proposition (Completing Markets) **Assume:** One SIR $X > 0$ and $(M-1)$ options on the SIR with strike prices $X_1, \dots, X_{M-1}$
Payoff: $D = \begin{bmatrix} X_1 & 0 & 0 & \cdots & 0 \\ X_2 & X_2 - X_1 & 0 & \cdots & 0 \\ X_3 & X_3 - X_1 & X_3 - X_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ X_M & X_M - X_1 & X_M - X_2 & \cdots & X_M - X_{M-1} \end{bmatrix}$ full rank $\Rightarrow$ Complete Market

Example: Let $X_{\omega+1}$-$X_\omega = \delta > 0$: $D = \begin{bmatrix} \delta & 0 & \dots & 0 \\ 2\delta & \delta & \dots & 0 \\ \dots & \dots & \dots & \dots \\ M\delta & (M-1)\delta & \dots & \delta \end{bmatrix}$

Note: Get AD Security/State-ω contingent claim with butterfly (Λ):
Long 1 Call($(\omega-1)\delta$), Short 2 Calls ($\omega\delta$), Long 1 Call($(\omega+1)\delta$)
$\implies$ pays δ only in state ω: Payoff=$[0, \dots, \omega, \dots, 0]$
Get State prices & risk-neutral measure:
$\phi_\omega = \frac{1}{\delta}\left[(c(K_{\omega+1}) - c(K_\omega)) - (c(K_\omega) - c(K_{\omega-1}))\right]$
Price of ANY security with payoff $\tilde{D}$:
$P_\omega = \sum_\omega \phi_\omega \tilde{D}_\omega \to \int_0^\infty \frac{\partial^2 c(K)}{\partial K^2}\tilde{D}(K)\, dK$
$\implies S = \int_0^\infty \frac{\partial^2 c(K)}{\partial K^2} X(K)\, dK$ and $B = \int_0^\infty \frac{\partial^2 c(K)}{\partial K^2} \cdot 1\, dK$
Recover $\mathbb{Q}$ measure: $q(K) = \frac{\partial^2 c(K)}{\partial K^2} / \int_0^\infty \frac{\partial^2 c(K)}{\partial K^2}\, dK$; VIX = STD($\mathbb{Q}$)

Exact Arbitrage Pricing Theory (APT)

Idea: Price redundant assets using set of prices of base securities.
$\implies$ Put constraints on SDF η: restrict co-movements of CFs

Beta/Expected Return Decomposition:

Theorem (FTAP Recall) Assume NO Redundant Asset:
NA $\Rightarrow \exists! \phi^* \in \mathbb{R}^M$ s.t. $P' = (\phi^*)'D$, where $\phi^* = D\theta$ for some $\theta \in \mathbb{R}^N$
Note: If rank $(D) = N$, then $D'D$ full rank: $P' = \theta' D' D$.
$\implies \theta^* = (D'D)^{-1}P$, $\phi^* = D(D'D)^{-1}P$ and $\eta^* = \phi^* \text{Diag}(p)^{-1}$

Proposition (SDF Mimicking Portfolio) Assume $\exists r^f$:
Construct portf mimicking η: $r_{-\eta} = \mathbb{E}^{\mathbb{P}}[r_{-\eta}] - (\eta - \mathbb{E}^{\mathbb{P}}[\eta]) = cst - \eta^*$
Furthermore, $\eta = \mathbb{E}^{\mathbb{P}}[r_{-\eta}] + \mathbb{E}^{\mathbb{P}}[\eta] - r_{-\eta}$
Note: $r_{-\eta}$ achieves H-J bound: maximum Sharpe Ratio!
($\because$) **(1)** Construct Portfolio with payoff η^*: Price $= -\sum_{\omega=1}^M \phi_\omega^* \eta_\omega^*$
(2) Want Payoff = Return i.e., investing 1\$ gives return $(1 + r_{-\eta})$:
$\longrightarrow$ Need to invest $1 + \sum_{\omega=1}^M \phi_\omega^* \eta_\omega^*$ in RF asset.

Proposition ($\mathbb{E}$[returns] Decomposition)
Fund. Asset Pricing Eq: $\pi_n = \mathbb{E}^{\mathbb{P}}\left[r_n - r^f\right] = -(1+r^f)\text{Cov}^{\mathbb{P}}\left(\eta, r_n - r^f\right)$

$\implies \pi_n = \mathbb{E}^{\mathbb{P}}\left[r_n - r^f\right] = \frac{\text{Cov}^{\mathbb{P}}(r_n, r_{-\eta})}{\text{Var}^{\mathbb{P}}(\eta)}\cdot\lambda = \beta_n\cdot\lambda$

with $\lambda := \mathbb{E}^{\mathbb{P}}[r_{-\eta}] - r^f = (1+r^f)\text{Var}^{\mathbb{P}}(\eta)$

$(\because)$ $\lambda := \mathbb{E}^{\mathbb{P}}[r_{-\eta}] - r^f = -(1+r^f)\text{Cov}^{\mathbb{P}}(\eta, -\eta) = (1+r^f)\text{Var}^{\mathbb{P}}(\eta)$

$\eta = \mathbb{E}^{\mathbb{P}}[r_{-\eta}] + \mathbb{E}^{\mathbb{P}}[\eta] - r_{-\eta} = (\lambda + r^f) + \frac{1}{1+r^f} - r_{-\eta}$

$\implies \eta = \lambda + \frac{1}{1+r^f} - (r_{-\eta} - r^f) \implies \pi_n = +(1+r^f)\text{Cov}^{\mathbb{P}}(r_{-\eta}, r_n)$

Definition (Market Price of Risk) λ = market $(r_{-\eta})$ risk premium
$\longrightarrow$ compensation an investor receives per unit of exposure to SDF

Definition (SDF Risk Exposure/Loading)
β_n = loading of asset n on $(r_{-\eta} - r^f)$

Prop: Asset n's expected return depends only on its loading β_n

Proposition (OLS Estimation of β_n) $\hat\beta_n = \frac{\widehat{\text{Cov}}(r_n, r_{-\eta})}{\widehat{\text{Var}}(r_{-\eta})}$

$(\because)$ Assume IID realizations of returns indexed by t. Do OLS on:
$r_{n,t} - r^f = \alpha_n + \beta_n(r_{-\eta,t} - r^f) + \varepsilon_{n,t}$ with $\mathbb{E}\left[\varepsilon_{n,t}|r_{-\eta} - r^f\right] = 0$

Definition (Idiosyncr Risk) $r_{n,t} - r^f = \alpha_n + \beta_n(r_{-\eta,t} - r^f) + \varepsilon_{n,t}$
= (abnormal ret) + (co-mvmt w/ SDF's excess ret $r_{-\eta}$) + (idiosyncr)

Proposition (Var Deco) $\text{Var}\left(r_n - r^f\right) = \beta_n^2\text{Var}\left(r_{-\eta} - r^f\right) + \text{Var}(\varepsilon_n)$
$\implies$ Variance = systematic/priced var + unpriced/residual var

Assumptions (Joint Hypoth) No Arb + Correct SDF model $r_{-\eta,t}$
$\implies$ Stocks w/ different $\varepsilon_{n,t}$ & same β_n earn same return $r_{n,t}$
$\implies$ $\alpha_n = 0$! If get $\alpha_n > 0$: reject BOTH hypoth: $\exists$ Arb OR bad SDF

Definition (Security Market Line - SML) Plot: $\mathbb{E}[r_n]$ vs. β_n
Slope = λ (market price of risk)

Factor Structure:

Assumptions M states, N securities w/ payoff matrix D
- $\exists$ RF security (assume $n = 1$ is the RF asset)
- $\text{rank}(D) = K \leq M \longrightarrow \exists$ redund assets

Definition (Factor Structure) F = Basis for D
$F = [F_1, \dots, F_K] \in \mathbb{R}^{M\times K}$, and $F_k \in \mathbb{R}^{M\times 1}$

Definition (Betas of Payoffs on Factors) $\beta_Z \in \mathbb{R}^K$
Payoff Space $C = \text{span}(D) = \left\{D\theta : \theta \in \mathbb{R}^N\right\}$
$\implies \forall Z \in C, \exists \beta_Z = [\beta_{Z1}, \dots, \beta_{ZK}]'$ s.t. $Z = F\beta_Z$
Any payoff $Z \in C$ can be spanned by the factors: $Z = F\beta_Z$
Conversely: Any factor can be replicated by securities in D: $F_k = D\theta_k$

Proposition (Factor Pricing) $\exists$ coeffs $\lambda = [\lambda_1, \dots, \lambda_K] \in \mathbb{R}^K$
s.t. $V(Z) = \lambda'\beta_Z = \sum_{k=1}^K \lambda_k\beta_{Zk}$, $\forall Z \in C$.
$(\because)$ NA $\implies \exists V(\cdot)$ linear: $V(Z) = V(\sum_k \beta_{Zk}F_k) = \sum_k \beta_{Zk}V(F_k)$
Note: $\lambda_k = -V(F_k)$ is INDEPENDENT of Z!
$\implies F_k$ = risk factors & λ_k = market price of risk

Definition $n = 1, \dots, N$ Securities:
- **Gross Return:** $R_n = D_n/P_n \in R^M$
- **Market Structure:** $R = [R_1, \dots, R_N] \in \mathbb{R}^{M\times N}$
- $\bar R_n = \mathbb{E}^{\mathbb{P}}[R_n] \in \mathbb{R}^M$, $\bar R = [\bar R_1, \dots, \bar R_N] \in \mathbb{R}^{M\times N}$
- $\varepsilon_n = R_n - \bar R_n \in \mathbb{R}^M$, $\varepsilon = [\varepsilon_1, \dots, \varepsilon_N] \in \mathbb{R}^{M\times N}$
- $\beta_n \in \mathbb{R}^K$, $\beta = [\beta_1, \dots, \beta_N] \in \mathbb{R}^{K\times N}$

Proposition (Factor Pricing for Returns) Assume $\mathbb{E}^{\mathbb{P}}[\varepsilon_n] = 0$:
$R_n = \bar R_n + \varepsilon_n = \bar R_n + F\beta_n = \bar R_n + \sum_{k=1}^K F_k\beta_{nk}$
$\implies R = \bar R + \varepsilon = \bar R + F\beta$ (so need $\mathbb{E}^{\mathbb{P}}[F\beta] = 0$)
$\longrightarrow$ One RF factor $\iota := F_1 = \mathbf{1}_M$ and $(K-1)$ risk factors

Example: (2-Securities) Gross returns:
Security 1: $1 \to \begin{cases} 1 \text{ w.p. } 0.5 \\ 1 \text{ w.p. } 0.5 \end{cases}$ Security 2: $1 \to \begin{cases} 1/2 \text{ w.p. } 0.5 \\ 2 \text{ w.p. } 0.5 \end{cases}$
$\bar R_1 = 0.5(1+1) = 1$ and $\bar R_2 = 0.5(1/2+2) = 1.25$
$R_n = \bar R_n + \varepsilon_n \implies \varepsilon_0 = [0,0]'$ and $\varepsilon_1 = [-0.75, 0.75]'$
Risk-free factor: $F_0 = [1,1]'$. Only 1 Risk factor: $\varepsilon_1 = F_1\cdot\beta_1$
$\implies F_1 = [-1,1]', \beta_1 = 0.75$

Lemma • The value of any gross return is 1: $V(R_n) = \frac{\mathbb{E}^{\mathbb{Q}}[R_n]}{R^f} = 1$
$(\because)$ Asset Payoff=D: $V(D) = \frac{\mathbb{E}^{\mathbb{Q}}[D]}{R^f} \implies V(R) = \frac{1}{R^f}\mathbb{E}^{\mathbb{Q}}\left[\frac{D}{V(D)}\right] = 1$
Note: I pay 1\$ today, get back R_n\$ tomorrow
- The value of the sure Gross Return is 1: $V(R^f) = V(1+r^f) = 1$
- Under $\mathbb{Q}$: $\mathbb{E}^{\mathbb{Q}}[R_n] = R^f \implies \mathbb{E}^{\mathbb{Q}}[R_n] - R^f = 0$

Theorem (Exact APT) Let $R_n = \bar R_n + F\beta_n$, $n = 1, \dots, N$, where:
(1) $F = [F_1, \dots, F_K]$ (K risk factors, $\mathbb{E}^{\mathbb{P}}[K] = 0$)
(2) $\beta_n = [\beta_{n1}, \dots, \beta_{nK}]'$ (asset n's beta)
N.A. $\implies \bar R_n - R^f = \bar r_n - r^f = \sum_{k=1}^K \lambda_k\beta_{nk} = \lambda'\beta_n$, $n = 1, \dots, N$
where $\lambda_k = -\mathbb{E}^{\mathbb{Q}}[F_k]$ and $\lambda = [\lambda_1, \dots, \lambda_K]'$
$(\because)$ $R^f = \mathbb{E}^{\mathbb{Q}}[R_n] = \mathbb{E}^{\mathbb{Q}}\left[\bar R_n + \sum_{k=1}^K F_k\beta_{nk}\right] = \bar R_n + \sum_{k=1}^K \beta_{nk}\mathbb{E}^{\mathbb{Q}}[F_k]$

Definition (Portfolio Beta) Given portfolio θ:
Its beta on risk Factor k: $\beta_k = \sum_{i=1}^K \theta_i\beta_{ik}$

Definition (Factor Mimicking Portfolio)
Portfolio θ_k s.t. $\beta_{ki} = \delta_{ki}$ $(i = 1, \dots, K)$
Prop: For each factor F_k, $\exists$ a factor portfolio θ_k

Corollary θ_k Factor Portfolio $\implies F_k = R_k - \bar R_k = r_k - \bar r_k$

Definition (Factor Premium) $\lambda = \bar R_k - R^f = \bar r_k - r^f$
$\implies$ Expected excess return on factor portfolio

Proposition $\bar r_n - r^f = \sum_{k=1}^K \lambda_k\beta_{nk} = \sum_{k=1}^K \beta_{nk}(\bar r_k - r^f)$,
where $\lambda_k = \bar r_k - r^f$: risk premium of k^{th} factor portfolio
$(\because)$ Under Exact APT, use factor portfolio

General Arbitrage Pricing Theory (APT)

Exact-APT Issue: Need Complete Market($K = M$) $\Rightarrow$ large #factors
Idea: We want:
- Model: large M (# states) & N (# assets) BUT small K (# factors)
- Study implications of No Asymptotic Arbitrage (NAA)

General Factor Model:

Assumptions (Factor Model for Returns) Suppose:
$r_n = \bar r_n + \sum_{k=1}^K \beta_{nk}F_k + \varepsilon_n$, for $n = 1, \dots, N$
with: **(1)** $\mathbb{E}^{\mathbb{P}}[F_k] = \mathbb{E}^{\mathbb{P}}[\varepsilon_n] = \mathbb{E}^{\mathbb{P}}[\varepsilon_n|F_k] = 0$, $\forall k, n$
(2) $\mathbb{E}^{\mathbb{P}}\left[\varepsilon_n^2\right] = \sigma_n^2 < v < \infty$, and $\mathbb{E}^{\mathbb{P}}[\varepsilon_n\varepsilon_{n'}] = 0$ $\forall n \neq n'$
Note: Exact Model: $\varepsilon_n = 0$ for all n

Note: (Matrix Notation) Let $r = \bar r + F\beta + \varepsilon$ where:
(1) $\mathbb{E}^{\mathbb{P}}[F] = \mathbb{E}^{\mathbb{P}}[\varepsilon] = \mathbb{E}^{\mathbb{P}}[\varepsilon|F] = 0$
(2) $\Sigma := \mathbb{E}^{\mathbb{P}}[\varepsilon'\varepsilon] = \text{Diag}(\sigma_1, \dots, \sigma_N)$
$r = [r_1, \dots, r_N] \in \mathbb{R}^{M\times N}$, $\bar r = [\bar r_1, \dots, \bar r_N] \in \mathbb{R}^{M\times N}$
$F = [F_1, \dots, F_K] \in \mathbb{R}^{M\times K}$, $\varepsilon = [\varepsilon_1, \dots, \varepsilon_N] \in \mathbb{R}^{M\times N}$
$\beta_n = [\beta_{n1}, \dots, \beta_{nK}]' \in \mathbb{R}^K$, $\beta = [\beta_1, \dots, \beta_N] \in \mathbb{R}^{K\times N}$

Proposition (Variance Decomposition) Under Current Model:
$\text{Var}^{\mathbb{P}}(r_n) = \beta_n'\mathbb{E}^{\mathbb{P}}\left[F'F\right]\beta_n + \text{Var}^{\mathbb{P}}(\varepsilon_n)$
$\text{Cov}^{\mathbb{P}}(r_i, r_j) = \beta_i'\mathbb{E}^{\mathbb{P}}\left[F'F\right]\beta_j$ for all $i \neq j$ $(\because)$ $\text{Cov}^{\mathbb{P}}(\varepsilon_i, \varepsilon_j) = 0$

Diversification:

Definition (Return on a Portfolio) $r_\theta = \bar r_\theta + F\beta_\theta + \varepsilon_\theta$
where: $r_\theta = \theta r$, $\bar r_\theta = \theta\bar r$, $\beta_\theta = \theta\beta$, $\varepsilon_\theta = \theta\varepsilon$

Definition (Well Diversified Portfolio) $\theta \in \mathbb{R}^N$ Well-Diversified if:
$\theta_n = O(1/n)$, where $\theta = [\theta_1, \dots, \theta_N]'$, $\theta'\mathbf{1}_N = \sum_{n=1}^N \theta_n = 1$.
Note: $\theta_n = O(1/n) \iff n\cdot\|\theta\|^\infty < \infty$

Definition (Well Diversified Sequence of Portfs)
$\{\theta_n\}_{n=1}^\infty$, with $\theta_n'\mathbf{1}_n = \sum_{i=1}^n \theta_{n,i} = 1$, is Well-Diversified
$\iff \exists\kappa \in (0,\infty)$, s.t. $\theta_{n,i}^2 < \kappa/n^2$, $\forall i = 1, \dots, n$, $\forall n \geq 1$
Note: $\theta_n = [\frac{1}{n}, \dots, \frac{1}{n}]'$=diversified, but $[0, \dots, 1, \dots, 0]'$=concentrated

Definition (Equally-Weighted Portfolio) $\theta = [\frac{1}{N}, \dots, \frac{1}{N}]'$
Prop: $\text{Var}^{\mathbb{P}}(\theta'r) = \text{Var}^{\mathbb{P}}\left(\sum_{n=1}^N \frac{1}{N}r_n\right) = \frac{1}{N}\left[\frac{1}{N}\sum_{n=1}^N \text{Var}^{\mathbb{P}}(r_n)\right]$
$+ \left(1 - \frac{1}{N}\right)\left[\frac{1}{N(N-1)}\sum_{n=1}^N\sum_{n'\neq n}\text{Cov}^{\mathbb{P}}(r_n, r_n')\right]$
If $r_n = \bar r + F + \varepsilon_n$, $\text{Var}^{\mathbb{P}}(\varepsilon_n) = \sigma_\varepsilon^2$:
$\implies \text{Var}^{\mathbb{P}}(\theta'r) = \frac{1}{N}\sigma_\varepsilon^2 + \left(1 - \frac{1}{N}\right)\text{Var}^{\mathbb{P}}(F) \longrightarrow \text{Var}^{\mathbb{P}}(F)$

Idea: Covariance with risk affects an asset's risk premium. Should only price systematic risk (explains returns' variation across all assets). But can $\exists$ non-priced syst risk (e.g. linear comb of βs not associated with changes in $\mathbb{E}$[ret], $\lambda_k = 0$ for some k in APT)ex: Risk Neutr ppl

Theorem (Diversification Thm) $\{\theta_n\}_{n=1}^\infty$ well div portfs seq:
$\text{Var}^{\mathbb{P}}(\varepsilon_{\theta_n}) = \text{Var}^{\mathbb{P}}\left(\sum_{i=1}^n \theta_{n,i}\varepsilon_i\right) \longrightarrow 0$ at rate $O(1/n)$ (i.e,$n\cdot\|\theta_i\| < C$)
Note: Well Div portfs have only systematic/factor risks (no idiosyncr)

General APT:

Definition (Asymptotic Arbitrage−AA) $\{\theta_n\}_{n=1}^\infty$ Portfs seq s.t.
(1) Self Financed: $\mathbf{1}_n'\theta_n = 0$
(2) $\mathbb{E}^{\mathbb{P}}[r_{\theta_n}] \longrightarrow \alpha > 0$
(3) $\text{Var}^{\mathbb{P}}(r_{\theta_n}) \longrightarrow 0$

Note: AA = arbitrage in the limit. For n finite, portf carries tiny risk Volatility may not be a sufficient measure of risk.

Prop: NAA $\implies$ NA ?

Theorem (General APT)
Given: K-Factors Model for security returns + NAA
$\implies \exists r^f \in \mathbb{R}, \lambda = [\lambda_1, \dots, \lambda_K]' \in \mathbb{R}^K$ s.t.:
$\sum_{i=1}^n\left[\bar r_i - (r^f + \lambda'\beta_i)\right]^2 = \sum_{i=1}^n\left[\bar r_i - \left(r^f + \sum_{k=1}^K \lambda_k\beta_{ik}\right)\right]^2 < A < \infty$
NAA $\implies$ approx. factor pricing: $\bar r_i - r^f \approx \sum_{k=1}^K \lambda_k\beta_{ik}$, $\forall i$
Note: So pricing error $\delta \neq 0$ only for small nb of assets: $\sum_i \delta_i^2 < A < \infty$
$(\because)$ 1 Factor: project $\bar r \in \mathbb{R}^n$ on $(\mathbf{1}_n, \beta) \longrightarrow \bar r = a_0\mathbf{1}_n + a_1\beta + \delta$
where $\delta \in \mathbb{R}^n$, $\delta'\mathbf{1}_n = 0$ (self-financing portf) and $\delta'\beta = 0$ (by $\perp$ proj.)
Note: $\beta_\delta = \delta'\beta = 0 \implies$ the δ portf has NO factor risk.
Take $\tilde\delta = b\delta$ for some $b > 0$: also self-financing
$\implies \mathbb{E}^{\mathbb{P}}\left[r_{\tilde\delta}\right] = \mathbb{E}^{\mathbb{P}}\left[b\delta'r\right] = b\delta'\bar r = b\delta'(a_0\mathbf{1}_n + a_1\beta + \delta) = b\delta'\delta = b\|\delta\|^2$
and $\text{Var}^{\mathbb{P}}\left(r_{\tilde\delta}\right) = \text{Var}^{\mathbb{P}}\left(b\delta'r\right) = b^2\delta'\Sigma_n\delta < b^2v\delta'\delta = b^2v\|\delta\|^2$
Fix $b = 1/\|\delta\|^2 \implies \mathbb{E}^{\mathbb{P}}\left[r_{\tilde\delta}\right] = 1$ & $\text{Var}^{\mathbb{P}}\left(r_{\tilde\delta}\right) < v/\|\delta\|^2 \to 0 \implies$ AA!

Corollary (General APT: Well-Diversified Case) NAA + Well Div
$\implies$ **Exact** factor pricing $\bar r_i - r^f = \sum_{k=1}^K \lambda_k\beta_{ik}$, $\forall i$

Corollary (General APT: Implications) For large N, Small K:
$\implies$ APT applies to most (not all) securities:
$\bar r_i - r^f = \sum_{k=1}^K \beta_{ik}(\bar r_k - r^f) = \sum_{k=1}^K \lambda_k\beta_{ik}$
where β_{ik} = factor loading of asset i on factor k
& $\lambda_k = \bar r_k - r^f$ = risk premium of factor k mimicking portfolio
$\implies$ "APT $\iff$ SDF model affine in factors: $\eta = a + Fb$"

Testing APT & Linear SDF Models:

Idea: Test if E[returns] lie on the SML implied by factor's E[return]
Data: (usually monthly) T-bill rates & asset + factor returns

Example: (Time-Series Approach)
GOAL: Regress asset's $r_n - r^f$ on factors: is $\alpha = 0$?
Assume: $\beta_{i,k}$ cst over time $\implies beta_{\text{portf}}$ more stable than β_{asset}
- Run times series regression for tradable asset/portfolio i:

$r_{i,t} - r_t^f = \alpha_i + \sum_{k=1}^K \beta_{i,k}(r_{k,t} - r_t^f) + \varepsilon_{i,t}$, for $t = 1\cdots T$
- Test if $\hat\alpha_i = 0$ or jointly test if $\{\hat\alpha_i\}_{i=1}^N = 0$

Example: (Cross-Sectional Approach)
GOAL: Make empirical counterpart to SML: β estimates explain $\bar r_n$?
Geometric Interpretation: Estimate the SML with linear regression
Step 1: estimate β for each asset through time-series regression:
$\implies r_{i,t} - r_t^f = \alpha_i + \sum_{k=1}^K \beta_{i,k}F_{k,t} + \varepsilon_{i,t} \longrightarrow$ save $\left\{\hat\alpha_i, \hat\beta_i\right\}_{i=1}^N$
Step 2: Use results to run cross-sectional regression:
$\implies \frac{1}{T}\sum_{t=1}^T(r_{i,t} - r_t^f) \xrightarrow{\text{i.p.}} \mathbb{E}\left[r_i - r^f\right] = \lambda_0 + \sum_{k=1}^K \lambda_k\hat\beta_{i,k} + u_i$
- Check if $\lambda_0 = 0$ and $\lambda_k = \mathbb{E}[F_{k,t}]$ if factors=tradable excess returns
- Perfect model + enough data $\implies u_i = 0$ $\forall i$ and $R^2 = 100\%$

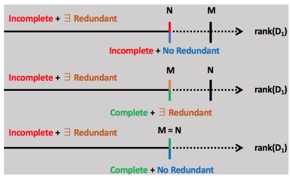
Note: Works with non-traded factors too: λ_0 unrestricted

Note: (Practical Considerations) Data $\implies$ get risk prices λ_k
$\implies$ Charactrz all major sources of syst risk + stock/portf's exposure
Step 1: Find data on factors: Returns (PCA), Fundam (macro data)
Step 2: Estimate β & λ: careful! missing factor? $\beta = \beta(t)$, $\lambda = \lambda(t)$?

Property (Fama-French) CAPM can't explain portf r_n sorted on simple firm characteristics (recently: momentum)
Beta: Sort returns vs $\beta \Rightarrow$ line too flat ($\beta_i = \text{Cov}(R_i, R_m)/\text{Var}(R_m)$)
Size: Small cap stocks returns > large cap stock returns
BV to Mrkt Ratio: (growth stck) $r_{\text{low ratio}} > r_{\text{high ratio}}$ (value stck)

In Practice

- **Payoff Space Spanned by Securities:** $C_1(D) = \{c_1 = D\theta : \theta \in \mathbb{R}^N\} = \text{span}(D_1, \ldots, D_N)$
- **Complete Market:** $\iff \text{span}(D) = \mathbb{R}^M \iff \text{rank}(D) = M$ $\iff \forall c_1 \in \mathbb{R}^M, \exists \theta \in \mathbb{R}^N$ S.T. $D\theta = c_1$
- **Arrow-Debreu Securities:** $D^{AD} := \mathbb{I}_{N \times N}$ (complete market): $\to$ construct using portfolio $\theta = D^{-1}$ $(\because)$ $D\theta = \mathbb{I} = D^{AD}$ $\to$ State Prices: $\phi = (D^{-1})'P$
- **Arbitrage Existence:** $\iff \exists \theta \in \mathbb{R}^N$ s.t. $B\theta > 0$.
  Type 1: $B\theta = [0, > 0]'$ **free** at $t = 0$, **maybe** paid at $t = 1$
  Type 2: $B\theta = [> 0, \geq 0]'$ **paid** at $t = 0$, **maybe** paid at $t = 1$
  Proposition Market Equilibrium $\implies \nexists$ arbitrage
  $(\because)$ invest ∞ amount in it $\implies$ no equilibrium
  Warning: $\nexists$ arbitrage $\not\Longrightarrow$ Market Equilibrium
- **State Price Vectors:** (consistent with NA) $P' = \Phi' D$.
- **Rank of $\mathbf{D_1}$:** $M = \#$states, $N = \#$assets, $D_1 = M \times N$.
  - rank $(D_1) < N$ $\Rightarrow \exists$ redundant
  - rank $(D_1) < M$ $\Rightarrow Im(D_1) \subsetneq \mathbb{R}^M$ Incompl
  - rank $(D_1) = N = M$ $\Rightarrow$Complete + $\nexists$ redundant



- **Feasible Set of Investor Choices:** $B(e_0, \{D_1, P\})$
  - **Complete, No redund:** rank $(D_1) = M = N$
    $\implies \theta = D^{-1}(c_1 - e_1)$
    $\implies B(e, \{D_1, P\}) = \left\{c_1 \in \mathbb{R}^M_+ : P'D_1^{-1}(c_1 - e_1) \leq e_0\right\}$.
  - **Complete & redund:** rank $(D_1) = M < N$
    $\implies \exists \tilde{D}_1(M \times M)$ full rank & $A(M \times (N - M))$ s.t. $D_1 = [\tilde{D}_1', (\tilde{D}_1 \times A)']'$
    $\implies (\tilde{c}_1 + \hat{c}_1) = (\tilde{e}_1 + \hat{e}_1) + \tilde{D}_1\tilde{\theta} + \tilde{D}_1 A\hat{\theta}$
    $\implies \tilde{\theta} = \tilde{D}_1^{-1}(\tilde{c}_1 - \tilde{e}_1)$ & $\hat{\theta} = (A'A)^{-1}A'\tilde{D}_1^{-1}(\hat{c}_1 - \hat{e}_1)$
    $$\implies B(e, \{D_1, P\}) = \Big\{\tilde{c}_1, \hat{c}_1 \in \mathbb{R}^M_+ : P'\tilde{D}_1^{-1}(\tilde{c}_1 - \tilde{e}_1) + P'(A'A)^{-1}A'\tilde{D}_1^{-1}(\hat{c}_1 - \hat{e}_1) \leq e_0; \; \hat{c}_1 = \hat{e}_1 + \tilde{D}_1 A(A'A)^{-1}A'\tilde{D}_1^{-1}(\hat{c}_1 - \hat{e}_1)\Big\}$$
  - **Incomplete, no redund:** rank $(D_1) = N < M$
    $\implies \exists \tilde{D}_1(N \times N)$ full rank & $\hat{D}_1((M - N) \times N)$ s.t. $D_1 = [\tilde{D}_1', \hat{D}_1']' \implies \theta = \tilde{D}_1^{-1}(\tilde{c}_1 - \tilde{e}_1)$
    $$\implies B(e, \{D_1, P\}) = \Big\{[\tilde{c}_1', \hat{c}_1'] \in \mathbb{R}^M_+ : P'\tilde{D}_1^{-1}(\tilde{c}_1 - \tilde{e}_1) \leq e_0; \; \hat{c}_1 = \hat{e}_1 + \hat{D}_1\tilde{D}_1^{-1}(\tilde{c}_1 - \tilde{e}_1)\Big\}$$
-

Optimal Portfolio Choices

Idea: **Before:** "reduced form" of course.
$\implies$ **Now:** build bottom-up micro-founded model

Expected Utility (E.U.) Theory

Preferences:

Idea: How agents use the fin market to best meet their econ needs

Definition (Consumpt° Set) $C = \{c = [c_0, c_1']' \in \mathbb{R}^{1+M}\} \subseteq \mathbb{R}^{1+M}$

Definition (Rational Preference)
Binary relat $\succcurlyeq^k$ over consump $C = \mathbb{R}^{1+M}_+$ s.t.
- **Complete:** $a, b \in C \implies a \succcurlyeq^k b$ or $b \succcurlyeq^k a$ or both.
- **Reflexive:** $a \in C \implies a \succcurlyeq^k a$
- **Transitive:** $a \succcurlyeq^k b$ & $b \succcurlyeq^k c \implies a \succcurlyeq^k c$

Axiom (Continuity) $\forall c \in C$: $\{a \in C : a \succcurlyeq c\}$ & $\{b \in C : b \preccurlyeq c\}$ closed
$\iff \forall \{a_n\} \to a, \{b_n\} \to b \in C : a_n \succcurlyeq b_n \Rightarrow a \succcurlyeq b$

Axiom (Insatiability) $a > b \implies a \succcurlyeq b$ (more $\succcurlyeq$ less)

Axiom (Convexity) $\forall a, b, c \in C$, $\forall \alpha \in (0, 1)$:
$a \gtrsim b$ & $c \succcurlyeq b \implies \alpha a + (1 - \alpha)c \succcurlyeq b$

Prop: Convex $\succcurlyeq \implies$ convex sets of preferred bundles $\{a \in C : a \succcurlyeq c\}$

Definition (Utility Function) $u : C \to \mathbb{R}$, s.t.
$a, b \in C : a \succcurlyeq b \iff u(a) \geq u(b)$.
Strictly Monotonic: $u(c) > u(c')$, $\forall c \succ c'$.
Prop: Insatiability $\implies$ u is strictly $\nearrow$, $u' > 0$
Indifference Curve: Plot u on a c_1 vs. c_0 plane

Theorem (Debreu) $C \subseteq R^{1+M}$ closed & cnvx, $\succcurlyeq$ rational + cont
$\implies \succcurlyeq$ can be represented by a continuous utility function u on C.

Definition (Expected Utility) over a consumption path/lottery
$u(c, p) = \sum_{\omega \in \Omega} p_\omega u_\omega(c_0, c_{1\omega})$, $\forall c \in C$.
Consumption Lotteries consumption in each state + probas (c, p)
Note: u depends on prob of future states
Upper-Contour Set: Indiff curve separates upper/lower region.
Von Neumann Morgenstern (vNM) utility: $u_\omega(c_0, c_1\omega)$
Prop: u & $u(c, p)$ are ordinal ; u_ω is cardinal!
Prop: u invariant w.r.t. $\nearrow$ transfos; u_ω invariant w.r.t. affine transfos

Axiom (Continuity) $\forall$ consumption $c \in C$, probas p_a, p_b, p_c:
$[c, p_a] \succcurlyeq [c, p_b] \succcurlyeq [c, p_c] \implies \exists \alpha \in (0, 1) : [c, p_b] \sim [c, (1 - \alpha)p_a + \alpha p_c]$

Axiom (Independence) $\forall$ consumption $c \in C$, $\alpha, p_a, p_b, p_c \in (0, 1)$:
$[c, p_a] \succcurlyeq [c, p_b] \implies [c, (1 - \alpha)p_a + \alpha p_c] \succcurlyeq [c, (1 - \alpha)p_b + \alpha p_c]$

Theorem (vNM, 1944) $\succcurlyeq$ on $(C, \mathbb{P})$ has a EU representation
$\iff \succcurlyeq$ rational + continuous + independent
$\implies u(c, p) = \sum_{\omega \in \Omega} p_\omega u_\omega(c_0, c_{1\omega})$

Assumptions (State Independence) $u_\omega(c_0, c_{1\omega}) = u(c_0, c_{1\omega})$

Assumptions (Time Additivity) $u(c_0, c_{1\omega}) = u(c_0) + \rho u(c_{1\omega})$
where $\rho \in (0, 1]$ is an (optional) time preference/discount coeff

Assumptions (No Complemnt$_+$/Substitute$_-$) $\frac{\partial^2 u_\omega(c_0, c_{1\omega})}{\partial c_0 \partial c_{1\omega}} = 0$

Assumptions (State Indep + Time Add for Exp. Util)
$u(c, p) = u(c_0) + \rho \sum_{\omega \in \Omega} p_\omega u(c_{1\omega})$ with $\rho \in (0, 1)$

Definition (Marginal Utility) At consumption level c: $u'(c)$
Prop: Insatiability $\implies u'(\cdot) > 0$ (so $u \uparrow$)

Definition (Concave Function)
$u(\alpha x + (1 - \alpha)x') \geq \alpha u(x) + (1 - \alpha)u(x')$
Prop: u concave & twice differentiable $\iff u'' \leq 0$, $u' \searrow$

Theorem (Concavity) $\succcurlyeq$ with Continuity + Indep + Convexity ax:
$\succcurlyeq$ can be represented by a discounted expected utility function
$u(c, p) = u(c_0) + \rho \sum_{\omega \in \Omega} p_\omega u(c_{1\omega}) \implies u(\cdot)$ concave
$(\because)$ Let consumpt° plan (c_0, c_1) with $c_{1\omega} = c_1$ (sure lottery at $t = 1$):
$u(c_0) + \rho u(c_1(c_0)) = \mu \implies u'(c_0) + \rho u'(c_1(c_0))c_1'(c_0) = 0$
$\implies u''(c_0) + \rho u''(c_1(c_0))c_1'(c_0)^2 = -\rho u'(c_1(c_0))c_1''(c_0)$
Note: $c_1(c_0)$ is LB of convex set in $\mathbb{R}^2 \implies c_1''(c_0) \geq 0$ (convex)
Along ray $c_0 = c_1 = c$: $u''(c)(1 + \rho c_1'(c)^2) = -\rho u'(c)c_1''(c) \leq 0$

Risk-Aversion

Definition (Fair Gamble) r.v. x s.t. $\mathbb{E}[x] = 0$
Definition (Risk Aversion) Agent with E.U. $u(\cdot)$ is risk-averse
$\iff \mathbb{E}[u(w + x)] \leq \mathbb{E}[u(w)]$ for any $\mathbb{E}[x] = 0$.
Note: RA $\implies$ Sure Payoff $\succcurlyeq$ Risky Payoff w/ Same Mean
Proposition (Concavity of u) Agent (strict) RA$\iff u$ (strict) concave
$(\because)$ $\boxed{\implies}$ $\forall w_1 < w_2, p \in (0, 1)$: Bern Gamble $x = \{x_1, x_2\}$ w.p. $(p, 1\text{-}p)$
s.t. $x_1 = -(1 - p)(w_2 - w_1)$ and $x_2 = p(w_2 - w_1) \implies \mathbb{E}[x] = 0$
Let $w = pw_1 + (1 - p)w_2$: so $w_1 = w + x_1$, $w_2 = w + x_2$
RA $\implies pu(w_1) + (1 - p)u(w_2) \leq u(w) = u(pw_1 + (1 - p)w_2)$ concave
$\boxed{\impliedby}$ u concave $\overset{\text{Jensen}}{\implies} \mathbb{E}[u(w + x)] \leq u(w + \mathbb{E}[x]) = u(w) \implies$ RA

Measures of Risk Aversion:

Definition (Risk Premium) x fair gamble, agnt w/ EU u, wealth w:
Risk Prem π required by agnt to take gamble: $\mathbb{E}[u(w + x)] = u(w - \pi)$
Note: π = amount of wealth an agent ok to give up to get rid of risk
Certainty Equivalent: $u(w_{CE}) = \mathbb{E}[u(w)]$ **Prop:** $\pi = \mathbb{E}[w] - w_{CE}$
$(\because)$ $u(w_{CE}) = \mathbb{E}[u(w)] = u(\mathbb{E}[w - \pi - x]) = u(w - \pi)$

Definition (Absolute Risk Aversion) $A(w) = -\frac{u''(w)}{u'(w)}$

Prop: Small Gamble x: $\pi \approx \frac{1}{2}A \cdot \text{Var}(x)$
$(\because)$ $\mathbb{E}[u(w + x)] = u(w) + \frac{1}{2}u''(w)\mathbb{E}[x^2] + o(x^2)$
$\overset{!}{=} u(w - \pi) = u(w)\text{-}u'(w)\pi + o(\pi)$
Note: $A(w)$ associated w/ risk premium per unit of absolute risk
Risk Tolerance: $T(w) = 1/A(w)$

Definition (Relative Risk Aversion) $R(w) = -w\frac{u''(w)}{u'(w)}$

Prop: Small Risk wx: $\pi_R \approx \frac{1}{2}R \cdot \text{Var}(x)$
$(\because)$ $\mathbb{E}[u(w(1 + x))] = u(w(1 - \pi_R))$
Note: risk premium $\propto R\times$ size of the risk (as a fract° of wealth)

Theorem (Pratt) Agents 1 & 2 w/ EU u_1 & u_2:
$A_1(w) \geq A_2(w)\ \forall w \iff u_1(u_2^{-1}(\cdot))$ concave
$\iff \exists f$ s.t: $f' > 0, f'' \leq 0$ & $u_1(w) = f(u_2(w))$
$\iff \pi_1 \geq \pi_2$, $\forall w$ & fair gambles x

$(\because)$ $f(z) = u_1(u_2^{-1}(z))$, $w = u_2^{-1}(z) \implies f(z) = \frac{u_1' u_2^{-1}}{u_2' u_2^{-1}}(z) > 0$

$\boxed{(1) \implies (2)}$ $f''(z) = -[A_1(w) - A_2(w)]\frac{u_1'(w)}{u_2'(w)^2} \leq 0$ for $A_1 \geq A_2$
$\boxed{(2) \implies (3)}$ Take $f(z) = u_1(u_2^{-1}(z))$
$\boxed{(3) \implies (4)}$ $u_1(w - \pi_1) = \mathbb{E}[u_1(w + x)] = \mathbb{E}[f(u_2(w + x))]$
[Jensen: f concave] $\leq f(\mathbb{E}[u_2(w + x)]) = f(u_2(w - \pi_2))$
$\boxed{(4) \implies (1)}$ Small gambles x: $\pi \propto A$ so trivial. Large gambles: Paper!

Examples of Risk Aversion:

Definition (CARA) Constant Absolute RA: $A'(w) = 0$
Definition (IARA/DARA) Incr/Decr Absolute RA: $A'(w) \gtrless 0$
Definition (CRRA) Constant Relative RA: $R'(w) = 0$
Definition (IRRA/DRRA) Incr/Decr Relative RA: $R'(w) \gtrless 0$
Example: (Linear EU) $u(w) = w$
$\implies$ **Risk Neutral** agents: $A(w) = R(w) = 0$
Example: (Negative Exponential EU) $u(w) = -e^{-aw}$, $a > 0$
$\implies$ **CARA** agents: $A(w) = a$, $R(w) = aw$
Example: (Quadratic EU) $u(w) = w - 0.5aw^2$, $a > 0$, $w \in [0, 1/a]$
$\implies$ **IARA** agents: $A(w) = \frac{a}{1 - aw}$, $R(w) = \frac{aw}{1 - aw}$
Example: (Log EU) $u(w) = \log w$
$\implies$ **CRRA** agents: $A(w) = 1/w$, $R(w) = 1$
Example: (Power EU) $u(w) = \frac{1}{1 - \gamma}w^{1-\gamma}$, $\gamma > 1$
$\implies$ **CRRA** agents: $A(w) = \gamma/w$, $R(w) = \gamma$ **Prop:** $\gamma \to 1 \Rightarrow$ Log EU

Example: (H$_{\text{yperbolic}}$ARA EU) $u(w) = a + b\left(d + \frac{w}{\gamma}\right)^{1-\gamma}$
$\implies$ **HARA** agents: $A(w) = \frac{1}{d + w/\gamma}$, $R(w) = \frac{w}{d + w/\gamma}$, $T(w) = d + \frac{w}{\gamma}$
Prop: Risk Neutral ($d = -\infty$), Quadr ($\gamma = -1$),
Neg Exp ($\gamma \to \infty, d = \frac{1}{a}$), Log ($d = 0, \gamma = 1$), Power ($d = 0, \gamma < 1$)

Optimal Consumption/Portfolio Choice

Assumptions (Setting) N non-redund assets, payoff D, price P
Agent: Endowt $e = [e_0, e_1']'$, Consumpt plan $c = [c_0, c_1']'$, Portf θ
EU: $u(c) = u_0(c_0) + \mathbb{E}[u_1(c_1)]$ with $u_t' > 0$, $u_t'' < 0$ $(t = 0, 1)$
Inada Condition: $\lim_{c \to 0} u_t'(c) = \infty$ (no need to assume $c \geq 0$)

Proposition (Agent's Optimization Pb) $\max_\theta u_0(c_0) + \mathbb{E}[u_1(c_1)]$
$[P'\theta = e_0 - c_0$=time-0 savings] s.t. $c_0 = e_0 - P'\theta$
$c_1 = e_1 + D\theta$

Theorem (Existence of Optimal Portf) Agent Optimization Pb:
$\exists$ solution $\iff$ No Arb in market $\{D, P\}$
$(\because)$ $\boxed{\Longrightarrow}$ If $\exists$Arb: agent can achieve unbounded consumpt° levels.
$\boxed{\Longleftarrow}$ If $\nexists$Arb: $\exists \phi \gg 0$ s.t. $P' = \phi' D$
Consumption financed by θ is $[-P'\theta, (D\theta)']'$
Agent's Budget: $B(e) = \left\{c \geq 0 : c = e + [-\phi' D\theta, (D\theta)']', \theta \in \mathbb{R}^N\right\}$
Use: $\hat{B}(e) = \left\{c \geq 0 : c = e + [-\phi' d, d']', d \in \mathbb{R}^M\right\}$ (with $N \leq M$)
Note: $B(e) = \left\{c \in \hat{B}(e) : d = D\theta\right\} \subseteq \hat{B}(e)$, $B(e) = \hat{B}(e) \Leftrightarrow M = N$
Now: $\hat{B}(e)$ bdd for $\phi \gg 0 \implies B(e)$ bdd + closed $\implies B(e)$ compact
u, u_0, u_1 continuous over compact $B(e) \implies$ max exists

Special Case: Complete Markets:

Assumptions
Complete set of AD securities, State Price $\phi \gg 0$
Agent: endowement $e = [e_0, e_1']'$, wealth $w = e_0 + \phi' e_1$
Budget: $B(e) = \{c : c_0 + \phi' c_1 = w\}$ (Simplify: ignore $c \geq 0$)
Marginal cost $= \phi_\omega$: Additional \$1 in asset $\omega \implies c_{1\omega} \nearrow$ by $1/\phi_\omega$

Proposition (Optimization) $\max_{c_0 + \phi' c_1 = w} u_0(c_0) + \sum_\omega p_\omega u_1(c_{1\omega})$

Lagrang: $\mathcal{L} = u_0(c_0) + \sum_\omega p_\omega u_1(c_{1\omega}) - \lambda[c_0 + \phi' c_1 - w] \to \partial_{c_0}, \partial_{c_1}$
FOC: $\lambda = u_0'(c_0) \longrightarrow$ marginal value of wealth
$\lambda \phi_\omega = p_\omega u_1'(c_{1\omega}) = \frac{\partial \mathbb{E}[u_1(c_1)]}{\partial \theta_\omega} \longrightarrow$ margin benefit of $\nearrow c_{1\omega} = \theta_\omega D_{1\omega}$
Note: At optim: relative marg utils for consumpt° in diff dates/states = their relative prices
$\eta_\omega = \frac{\phi_\omega}{p_\omega} = \frac{u_1'(c_{1\omega})}{u_0'(c_0)} =$ intertemp marg rate of substitution, $\frac{\phi_\omega}{\phi_{\omega'}} = \frac{p_\omega u_1'(c_{1\omega})}{p_{\omega'} u_1'(c_{1\omega'})}$

Proposition u_t strictly concave
$\implies u_t'$ strictly $\searrow$ & $u_t'^{-1}$ exists

Theorem (Optimal Portfolio Choice) Solve FOC:
$c_0 = u_0'^{-1}(\lambda)$ and $c_{1\omega} = u_1'^{-1}\left(\lambda \frac{\phi_\omega}{p_\omega}\right)$ $\forall \omega \in \Omega$
where λ solves budget constraint: $w = e_0 + \phi' e_1 = c_0(\lambda) + \phi' c_1(\lambda)$

Theorem Complete Market, agnts w/ insatiable + strictly concave EU:
$c_{1\omega} < c_{1\omega'} \iff \frac{\phi_\omega}{p_\omega} > \frac{\phi_{\omega'}}{p_{\omega'}}$ (for all $\omega, \omega' \in \Omega$, $\forall k$)
Note: At optimum: levels of consumption in diff states are ranked inversely by SPD $\eta \longrightarrow$ High pain $\eta_\omega \Rightarrow$ Low consumpt° $c_{1\omega}$

General Equilibrium: Lucas Tree Model (1978):

Assumptions Agents w/ identical prefs + endowments
- Complete Market: agents can freely trade resources over time/states
- Market Clearing: aggreg consumpt° $\sum_k c_k = \sum_k e_k$ aggreg endow

PROBLEM: Find Equilibrium State Prices + Risk-Free Rate
Prop: FOC + Market Clearing $\implies \frac{u_1'(c_{1\omega})}{u_0'(c_0)} = \frac{u_1'(e_{1\omega})}{e_0'(c_0)} = \frac{\phi_\omega}{p_\omega} = \eta_\omega$

Note: Denom known $\implies$ randomness in η_ω depends on $u'(c_{1\omega})$
$u'(c_{1\omega}) \searrow$ w/ $c_1\omega$ so: High Pain $\Rightarrow$ High Marg Util $\Rightarrow$ Low $c_{1\omega} = e_{1\omega}$
Example: (Special Case) $e_{1\omega} = e_1$ & $u_1(c) = \delta u_0(c) =: \delta u(c)$
$\implies$ constant SPD $\eta_\omega = \frac{\delta u'(c_{1\omega})}{u'(c_0)} =: \frac{1}{1+r^f}$
So: $1 + r^f > \frac{1}{\delta} \iff c_1 > c_0$ (ppl prefer to smooth consumption out)
Note: cst aggregate consumption $\implies 1 + r^f = 1/\delta$
No uncertainty: high growth + abundant resources $\Rightarrow$ interest rates$< \frac{1}{\delta}$

Example: (CRRA) $e_{1\omega} := \bar{c}_1 + \varepsilon_\omega$, $\mathbb{E}[\varepsilon_\omega] = 0$:
$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \implies u'(c) = c^\gamma \implies u''(c) = -\gamma c^{-\gamma-1}$
$\implies u'''(c) = \gamma(1+\gamma)c^{-\gamma-2} > 0$ so convex marginal util u'
$\overset{\text{Jensen}}{\implies} \frac{1}{1+r^f} = \mathbb{E}[\eta_\omega] = \mathbb{E}\left[\frac{\delta u'(c_{1\omega})}{u'(c_0)}\right] \geq \frac{\delta u'(\mathbb{E}[c_{1\omega}])}{u'(c_0)} = \frac{\delta u'(\bar{c}_1)}{u'(c_0)}$
Precautionary Savings Effect: Possibility of high marg util states in future makes agnt want to save more.
$\implies$ Pushes RF bond price $\nearrow$ and the $r^f \searrow$

Characterization of Optimal Portfolio:

Proposition (Optimization) Use Budget Constraint for c_0, c_1
$\implies \max_\theta u_0(e_0 - P'\theta) + \mathbb{E}[u_1(e_1 + D\theta)]$
$= \max_\theta u_0(e_0 - P'\theta) + \sum_\omega p_\omega u_1\left(e_{1\omega} + \sum_{n=1}^N \theta_n D_{1\omega n}\right)$

Definition (Euler Eqn) FOC: $u_0'(c_0)P_n = \mathbb{E}[u_1'(c_1)D_n]$, $n = 1..N$
At optimum: MU(t=0 consumpt loss: paid P_n to invest in 1 asset n) = MU(t=1 consumpt gain: receive payoff D_n from investment in asset n)
OR: $1 = \mathbb{E}\left[\frac{u_1'(c_1)}{u_0'(c_0)} \frac{D_n}{P_n}\right] =: \mathbb{E}\left[\frac{u_1'(c_1)}{u_0'(c_0)} R_n\right]$
MU(invest in traded assets)/MU(consuming today) = 1 $\forall n$
Note: FOC does not guarantee optimality: need **SOC**!

Definition (SOC) Optimality obtained if, in addition to FOC:
$u_0''(c_0)P_n^2 + \mathbb{E}[u_1''(c_1)D_n^2] \leq 0$, $n = 1, \ldots, N$
Prop: u_1, u_2 concave $\implies$ SOC holds

Proposition (Portfolio Decomposition)
Agent's $t = 0$ savings: $w = e_0 - c_0 = P'\theta$
$\implies$ Optimal consumpt/portf choice: $\max_w \{u_0(e_0 - w) + v_1(w)\}$
v function: $v_1(w) = \max_{\{\theta : P'\theta = w\}} \mathbb{E}[u_1(e_1 + D\theta)]$

Note: v =portf choice problem given total amount to invest $=w$
Example: (Special Case) $e_1 = 0$ (agent endowed only with e_0 cash)
$\implies$ Portf Choice Pb: $v(w) = \max_{\{\theta : P'\theta = w\}} \mathbb{E}[u_1(D\theta)]$

Riskless asset: asset N with gross return $R_n = 1 + r^f$
$a_n = \theta_n P_n$: \$ invested in asset $n \implies w = \sum_n a_n$ **total investment**
Portf payoff: $\tilde{w} = D\theta = \sum_{n=1}^N a_n R_n = w(1 + r^f) + \sum_{n=1}^{N-1} a_n(r_n - r^f)$
Excess Return of asset n: $r_n - r^f$
Theorem (General Pb)
$r = [r_1, \ldots, r_{N-1}]'$ returns on risky assets;
$a = [a_1, \ldots, a_{N-1}]'$ investments in risky assets;
$\implies$ Optimal Portfolio Pb:
$\max_a \mathbb{E}[u(\tilde{w})] = \max_a \mathbb{E}\left[u\left(w(1 + r^f) + (r - r^f \iota')a\right)\right]$
$\implies$ FOC: $\mathbb{E}\left[u'(\tilde{w})(r_n - r^f)\right] = 0$ $\forall n = 1..N-1$
$\longrightarrow$ Solution $= a(w) \in \mathbb{R}^{N-1}$
Note: FOC: marg cost of investing in n
$=\mathbb{E}[u'(\tilde{w})r_n] = \mathbb{E}\left[u'(\tilde{w})r^f\right]$=marg cost of losing r^f

Properties of Optimal Portfolio:

Case 1: Assume only **ONE** risky asset:
Prop: $\tilde{w} = w(1 + r^f) + a(r - r^f)$
$(\because)$ borrow at RF rate, invest in risky asset
Proposition (Opt Investment a) Agent = strictly RA
$a > 0 \iff \bar{r} > r^f$; $a < 0 \iff \bar{r} < r^f$
$a = 0 \iff r = r^f$
$(\because)$ $\bar{u}(a) = \mathbb{E}[u(\tilde{w})] = \mathbb{E}\left[u(w(1 + r^f) + a(r - r^f))\right]$
$\bar{u}''(a) = \mathbb{E}\left[u''(\tilde{w})(r - r^f)^2\right] \leq 0$ as u concave
$\implies$ at max: $0 = \bar{u}'(a) = \mathbb{E}\left[u'(\tilde{w})(r - r^f)\right]$
$\bar{u}'(0) = u(w(1 + r^f)) \cdot (\bar{r} - r^f) \to \text{sign}(r - r^f)$
$a < 0 \iff \bar{u}'(0) < 0 \iff r < r^f$
Prop: risk-premium$> 0 \implies$ agent invest at least ε in risky asset
$(\because)$ Change $a = 0$ to $a = \varepsilon$ small: $\frac{d\mathbb{E}[\tilde{w}]}{da} = \bar{r} - r^f$ indep of a
$\frac{d\text{Var}(\tilde{w})}{da} = \frac{d}{da}(a^2 \text{Var}(r)) = 2a\text{Var}(r) \nearrow$ with a

Proposition (Abs RA) Assume $\bar{r} - r^f > 0$ (so $a > 0$)
$a'(w) > 0 \Leftrightarrow A'(w) < 0$ (DARA) ; $a'(w) = 0 \Leftrightarrow A'(w) = 0$ (CARA)
$a'(w) < 0 \Leftrightarrow A'(w) > 0$ (IARA $\longrightarrow$ very rare)
$(\because)$ Consider DARA: $A'(w) < 0$. FOC: $\mathbb{E}\left[u'(\tilde{w})(r - r^f)\right] = 0$
$\implies \frac{d}{dw}$ and algebra: use $u'' < 0$, $a > 0$, $A(\tilde{w}) = -\frac{u''(\tilde{w})}{u(\tilde{w})}$
Prop: You see from FOC diff: $\frac{da}{dw} = -(1 + r^f) \frac{\mathbb{E}[u''(\tilde{w})(r - r^f)]}{\mathbb{E}[u''(\tilde{w})(r - r^f)^2]}$

Definition (Relative Propensity) for investor in risky asset:
$e(w) = \frac{w}{a} \frac{da}{dw}$
Note: $e(w) = 1 \Leftrightarrow a(w) = \bar{a} \cdot w$: risky invest^M = CST fract° of wealth
Proposition (Rel RA) Assume $\bar{r} - r^f > 0$ (so $a > 0$)
$e(w) > 1 \Leftrightarrow R'(w) < 0$ (DRRA) ; $e(w) = 1 \Leftrightarrow R'(w) = 0$ (CRRA)
$e(w) < 1 \Leftrightarrow R'(w) > 0$ (IRRA $\longrightarrow$ very rare)
$(\because)$ FOC diff $\implies e(w) = \frac{w}{a} \frac{da}{dw} = -\frac{w(1 + r^f)}{a} \frac{\mathbb{E}[u''(\tilde{w})(r - r^f)]}{\mathbb{E}[u''(\tilde{w})(r - r^f)^2]}$
$\implies e(w) - 1 = -\frac{1}{a} \frac{\mathbb{E}[u''(\tilde{w})(r - r^f)]}{\mathbb{E}[u''(\tilde{w})(r - r^f)^2]} = -\frac{1}{a} \frac{\mathbb{E}[R(\tilde{w})(-u'(\tilde{w}))(r - r^f)]}{\mathbb{E}[u''(\tilde{w})(r - r^f)^2]}$

Case 2: Assume **MULTIPLE** risky assets:
Prop: $\tilde{w} = w(1 + r^f) + (r - r^f \iota')$
Theorem (Opt Investment a) $a = 0 \Leftrightarrow \mathbb{E}[r_n] = r^f$ $\forall n = 1..N-1$
$(\because)$ $\Rightarrow$: Use FOC
$\Leftarrow$: risk prem = 0 for all risky assets
$\implies \mathbb{E}[\tilde{w}] = w(1 + r^f) \longrightarrow$ payoff from a=0 portf
Jensen: $\mathbb{E}[u(\tilde{w})] \leq u(\mathbb{E}[\tilde{w}]) = u(w(1 + r^f))$ for all $a \implies a = 0$ opt
Theorem (Opt Investment a II) Some risk-prem on risky assets$\neq 0$
$\implies \mathbb{E}[r_{\text{portf}}] > r^f$ (i.e., $\sum_{n=1}^{N-1} a_n(\mathbb{E}[r_n] - r^f) \geq 0$)
$(\because)$ Jensen: $u(\mathbb{E}[\tilde{w}]) \geq \mathbb{E}[u(\tilde{w})] \geq u(w(1 + r^f)) \implies \mathbb{E}[\tilde{w}] \geq w(1 + r^f)$
$\implies \sum_{n=1}^{N-1} a_n(\mathbb{E}[r_n] - r^f) \geq 0$

Stochastic Dominance

Idea: 2 key elements to rank portfs: $\mathbb{E}$[return] & risk $\longrightarrow$ tradeoff!
Use: partial order (returns props let agnts rank 2 portfs, indep of prefs)
Let r_A, r_B = returns of assets A & B

First Order Stochastic Dominance: (dominance in return distrib)

Definition (FSD) A dominates B in the FSD sense:
$A \gtrsim_{\text{FSD}} B \iff \forall u' \geq 0 : \mathbb{E}[u(r_A)] \geq \mathbb{E}[u(r_B)]$
Note: $u(r) = u(w(1 + r))$
Prop: $A \gtrsim_{\text{FSD}} B \implies \bar{r}_A \geq \bar{r}_B$ but converse FALSE!
Theorem F_A, F_B CDFs of R_A & R_B:
$A \gtrsim_{\text{FSD}} B \iff F_A(x) \leq F_B(x)$ $\forall x$
$\iff r_A \overset{d}{\sim} r_B + \varepsilon$, with $\varepsilon \geq 0$
$(\because)$ use IPP + recall $\mathbb{E}[u(R)] = \int_0^{\bar{R}} u(x)dF(x)$
Theorem (Ordering) $A \gtrsim_{\text{FSD}} B \implies$: for $u' > 0, u'' < 0$
$\max_a \mathbb{E}\left[u(w(1 + r^f) + a(r_A - r^f))\right] \geq \max_a \mathbb{E}\left[u(w(1 + r^f) + a(r_B - r^f))\right]$
$(\because)$ Let $f(a, r_i) := \mathbb{E}\left[u(w(1 + r^f) + a(r_i - r^f))\right]$ and
$a_B = \text{argmax}_a f(a, r_B)$: $\max_a f(a, r_A) \geq f(a_B, r_A) \geq \max_a f(a, r_B)$

Second Order Stochastic Dominance: (dominance in risk)

Definition (SSD) A dominates B in the SSD sense:
$A \gtrsim_{\text{SSD}} B \iff \forall u'' \leq 0 : \mathbb{E}[u(R_A)] \geq \mathbb{E}[u(R_B)]$
Prop: ONLY WORKS IF $\bar{R}_A = \bar{R}_B$!
Prop: $A \gtrsim_{\text{SSD}} B \implies \text{Var}(R_A) \leq \text{Var}(R_B)$ but converse FALSE!
Theorem (Rothschild-Stiglitz) $A \gtrsim_{\text{SSD}} B$
$\iff \mathbb{E}[R_A] = \mathbb{E}[R_B]$ and $\int_0^y [F_A(x) - F_B(x)]dx =: S(y) \leq 0$ $\forall y$
$\iff R_A \overset{d}{\sim} R_B + \varepsilon$, with $\mathbb{E}[\varepsilon | R_B] = 0$
Prop: $R_A \sim N(\mu, \sigma_A^2)$, $R_B \sim N(\mu, \sigma_B^2)$: $\sigma_A < \sigma_B \implies A \gtrsim_{\text{SSD}} B$
$(\because)$ $R_B \overset{d}{\sim} R_A + \varepsilon \Rightarrow \varepsilon \sim N(0, \sigma_B^2 - \sigma_A^2) \perp\!\!\!\perp B \Rightarrow \mathbb{E}[\varepsilon | R_A] = 0$
Note: $\text{Var}(R_A) \leq \text{Var}(R_B) \not\Rightarrow A \gtrsim_{\text{SSD}} B$: try utility that has small $\mathbb{P}$ of black swan

Mutual-Fund Separation Thms

Idea: Characterize Opt Portfs: Explore restrictions on
(**1**) Return Distributions (**2**) Agent's Utility Function (**3**) Both

Assumptions (Setup)
$x_n = \frac{a_n}{w}$: weight of portf a in asset n ($n = 1..N$)
$x = [x_1, \dots, x_N]' \in \mathbb{R}^N$ with $\iota' x = 1$
Note: x defines a portf with gross return $R_x = Rx$

Proposition (Optimal Portf Pb) v =util funct° over returns
$\max_{\{x:\iota'x=1\}} \mathbb{E}[u(\tilde{w})] = \max_{\{x:\iota'x=1\}} \mathbb{E}[u(wRx)] = \max_{\{x:\iota'x=1\}} \mathbb{E}[v(Rx)]$

Definition (F-Fund Separation) The set of optimal portfs of different agents lie in a $F-1 \le N-1$ dimens affine subspace of $\mathbb{R}^N$
X_F= the $F-1$ dimensional subspace
$x_k \in \mathbb{R}^N$= agent k's opt portf ($k = 1..K$)
$x_{Fi} \in \mathbb{R}$= F indep portfs/funds in X_F ($i = 1..F$)
F funds are called the **separating funds** or **mutual funds**
Prop: F-Fund separation $\implies x_k = \sum_{i=1}^F h_{ki} x_{Fi}$ with $\sum_{i=1}^F h_{ki} = 1$
Every agent's opt portf = lin comb of the F funds
Note: If $\exists$ a RF asset: it's often $\in F$ funds $\to$ **Monetary Separat°**
$\implies$ Remaining $F-1$ funds consist of only risky assets

Theorem (Ross - 2 Fund Sep)
Suppose: $\exists$ RF asset w/ gross return R^f (call it asset N).
2 Funds Separation holds for any insatiable & concave util funct°
$\iff$ returns on risky assets satisfy: $\exists F, \beta_n, x_n, n = 1..N-1$ s.t.
(**1**) $R_n = R^f + \beta_n F + \varepsilon_n$
(**2**) $\mathbb{E}[\varepsilon_n | F] = 0$
(**3**) $\sum_{n=1}^{N-1} x_n \varepsilon_n = 0$ where $\sum_{n=1}^{N-1} x_n = 1$
Note: RF security = a separating fund
Note: Returns of risky assets have a 1-factor structure
Note: $\exists$ a portf of only risky assets (w/ x_n=weight in risky asset $n \in \{1..N-1\}$) s.t. idiosyncratic risks are fully eliminated
portf with only factor risk $\gtrsim_{SSD}$ portf w/ factor + idiosyncr risk

Theorem (Preferences & Mutual-Fund Separation)
Assume: arbitrary return distrib & $u' > 0, u'' < 0$
1-Fund Sep holds $\iff$ all agents have the same util funct° over returns (up to affine transfos)
Note: 1-Fund Sep is STRONG: need $\sim$ identical prefs!

Example: (Homothetic CRRA) $u(wRx) = w^{1-\gamma} \cdot u(Rx)$
$\implies \text{argmax}_x \mathbb{E}[u(wRx)] = \text{argmax}_x \mathbb{E}[u(Rx)]$
Note: opt portf x indep of wealth level: can allow for heterogen in w
($\because$) $\implies \text{argmax}_x \mathbb{E}[u(wRx)] = \text{argmax}_x w^{1-\gamma}\mathbb{E}[u(Rx)]$

Theorem (Cass-Stiglitz) 2-Fund monetary sep holds
$\iff u_k'(w) = \left(d_k + \frac{w}{\gamma}\right)^{-\gamma} > 0$ for every agent k
with $d_k \ge 0$ and $\gamma \ge -1$ same for all agents

Mean-Variance Portfolio Theory

Idea: To characterize optimal portfolios, impose restriction on: Preferences (U) and/or Returns distribution
Here: focus on case where preferences over portfolio return is a function of mean + variance (i.e., suff stats for returns distrib)

Mean-Variance Preference:

Definition (Portfolio Choice Pb) $\max_{a:\iota' a=1} \mathbb{E}[u(\tilde{w})]$
where: a = portf weights, $\tilde{w}$ = total wealth
Prop: $\mathbb{E}[u(\tilde{w})] = \sum_n \frac{1}{n!} u^{(n)}(\mathbb{E}[\tilde{w}]) \cdot \mathbb{E}[(\tilde{w} - \mathbb{E}[\tilde{w}])^n]$
Idea: Focus on E.U. that depends on $\mathbb{E}[\tilde{w}]$ & $\text{Var}(\tilde{w})$

Definition (Mean-Variance Preferences)
$\mathbb{E}[u(\tilde{w})] = v(\mathbb{E}[\tilde{w}], \text{Var}(\tilde{w})) = v(\bar{w}, \sigma_w^2)$

Definition (Quadratic Utility) $u(w) = w - \frac{1}{2} a w^2$, $a > 0, w < 1/a$
Prop: $u(\cdot)$ quadratic $\implies \mathbb{E}[u(\tilde{w})] = v(\bar{w}, \sigma_w^2)$
($\because$) $\mathbb{E}[u(\tilde{w})] = \mathbb{E}[\tilde{w} - \frac{1}{2} a \tilde{w}^2] = \bar{w} - \frac{1}{2} a \bar{w}^2 - \frac{1}{2} a \sigma_w^2$
Prop: $v \nearrow \bar{w}, \searrow \sigma_w$ and is concave in $\bar{w}$ & σ_w
($\because$) $\frac{\partial v}{\partial \bar{w}} = 1 - a\bar{w} > 0$, $\frac{\partial v}{\partial \sigma_w} = -a\sigma_w < 0$,
and $\frac{\partial^2 v}{\partial \bar{w}^2} = \frac{\partial^2 v}{\partial \sigma_w^2} = -a < 0$, $\frac{\partial^2 v}{\partial \bar{w} \partial \sigma_w} = 0$

Proposition (Jointly Normal Returns) D jointly normal
$\implies \tilde{w} = D\theta \sim N(\bar{w}, \sigma_w)$
$\implies$ mean-variance preferences: $\mathbb{E}[u(\tilde{w})] = v(\bar{w}, \sigma_w^2)$
($\because$) $\tilde{w} = \bar{w} + \sigma_w \varepsilon$, where $\varepsilon \sim N(0, 1)$
Prop: $v(\bar{w}, \sigma_w) = \mathbb{E}[u(\tilde{w})]$, $u' > 0, u'' < 0$: $v \nearrow \bar{w}, \searrow \sigma_w$ + concave
($\because$) $\frac{\partial v}{\partial \bar{w}} = \mathbb{E}[u'(\bar{w} + \sigma_w \varepsilon)] > 0$,
$\frac{\partial v}{\partial \sigma_w} = \mathbb{E}[u'(\bar{w} + \sigma_w \varepsilon)\varepsilon] < \mathbb{E}[u'(\bar{w})\varepsilon] = 0$,
and $\frac{\partial^2 v}{\partial \bar{w}^2} = \mathbb{E}[u''(\bar{w} + \sigma_w \varepsilon)] < 0$, $\frac{\partial^2 v}{\partial \sigma_w^2} = \mathbb{E}[u''(\bar{w} + \sigma_w \varepsilon)\varepsilon^2] < 0$

Proposition (2nd Order Approx)
Risk Premium π: $\mathbb{E}[u(\tilde{w})] = \mathbb{E}[u(\bar{w} + \varepsilon)] = u(\bar{w} - \pi)$ with $\varepsilon = \tilde{w} - \bar{w}$
Small risk $\implies \pi(\bar{w}, \sigma_w^2) \approx \frac{1}{2}\left[-\frac{u''(\bar{w})}{u'(\bar{w})}\right]\sigma_w^2$
$\implies$ For small gambles: mean-variance prefs approximate any $u(\tilde{w})$

Mean-Variance Frontier Portfolios:

Idea: (2-Step Approach)
(**1**) Minimize $\text{Var}(\tilde{w})$ as a function of the a target $\mathbb{E}$[portf return] $\bar{r}_p$
$\implies$ Find set of **mean-variance frontier portfs**
(**2**) Pick portf $\bar{r}_p$ that maximizes $u(\tilde{w})$

Assumptions (Setup) $r = [r_1, \dots, r_N]' \in \mathbb{R}^N$ **asset returns** with:
mean $\bar{r} = \mathbb{E}[r]$ & **covariance matrix** $\Sigma = \mathbb{E}[(r - \bar{r})'(r - \bar{r})] \in \mathbb{R}^{N\times N}$
$x = [x_1, \dots, x_N] \in \mathbb{R}^N$ **portf weights**: $r_x = r'x$, $\bar{r}_x = \bar{r}'x$, $\sigma_x^2 = x'\Sigma x$
Terminal Wealth: $\tilde{w} = w(1 + r_x)$, $\mathbb{E}[\tilde{w}] = w(1 + \bar{r}_x)$, $\text{Var}(\tilde{w}) = w^2\sigma_x^2$
Agents prefer portfs w/ higher $\mathbb{E}[r_x]$ & lower σ_x^2.

Definition (Mean-Variance Frontier Portf − MVF) $\min_x \frac{1}{2} x'\Sigma x$
s.t. $\bar{r}'x = \bar{r}_p$ & $\iota' x = 1$

Proposition (Trick) $\frac{\partial}{\partial x} x'\Sigma x = \frac{\partial}{\partial x'} x'\Sigma x = x'(\Sigma + \Sigma') = 2x'\Sigma$

Assumptions (Additional) Only risky assets + No redundancy
$\implies \Sigma$ full rank & $\exists \Sigma^{-1}$

Theorem (MVF Solution) $\mathcal{L} = \frac{1}{2} x'\Sigma x + \lambda_1(\bar{r}_p - \bar{r}'x) + \lambda_2(1 - \iota' x)$
FOC: $x'\Sigma = \lambda_1 \bar{r}' + \lambda_2 \iota'$ with $\bar{r}'x = \bar{r}_p$ & $\iota' x = 1$
$\implies x = \lambda_1 \Sigma^{-1} \bar{r} + \lambda_2 \Sigma^{-1} \iota$ (plug-in constraints to get λ's)
Define: $x_1 := \frac{1}{\iota'\Sigma^{-1}\iota} \Sigma^{-1}\iota$ & $x_2 := \frac{1}{\iota'\Sigma^{-1}\bar{r}} \Sigma^{-1}\bar{r}$ **Prop:** $\iota' x_{1,2} = 1$
$\implies$ 2 frontier portfs with $\mathbb{E}$[returns] $\bar{r}_1 = \frac{\bar{r}'\Sigma^{-1}\iota}{\iota'\Sigma^{-1}\iota}$ & $\bar{r}_2 = \frac{\bar{r}'\Sigma^{-1}\bar{r}}{\iota'\Sigma^{-1}\bar{r}}$
$\implies x = \lambda x_2 + (1 - \lambda) x_1$
$\implies \bar{r}_p = \lambda \bar{r}_2 + (1 - \lambda) \bar{r}_1$ with $\lambda = \frac{\bar{r}_p - \bar{r}_1}{\bar{r}_2 - \bar{r}_1}$
Vary $\bar{r}_p$ to draw the Mean-Variance Frontier $\to$ only boundary!

Corollary (2-Funds Separation)
- Any MVP generated by mixing 2 MVF portfs: x_1 & x_2
- MVF generated by mixing any 2 MVF portfs
- Portfs of MVFs is an MVF Portf
- The set of MVF Portfs = **line** in $\mathbb{R}^N$

($\because$) Set of portfs: $X = \left\{x \in \mathbb{R}^N : \iota' x = 1\right\} \in \mathbb{R}^{N-1}$
Set of MVF portfs: $X_{MVF} \subseteq X$ and $x_1, x_2 \in X_{MVF} \implies$ line in $\mathbb{R}^N$

MVF Portfolio Properties:

Definition (Minimum Variance Portfolio − MVP) $\frac{\partial \sigma_p^2}{\partial \bar{r}_p} = 0$
Prop: Necessary & suff: $x_{MVP}'\Sigma(x_2 - x_1) = 0$
($\because$) $0 = \frac{\partial \sigma_p^2}{\partial \bar{r}_p} = \frac{\partial \sigma_p^2}{\partial x} \cdot \frac{\partial x}{\partial \bar{r}_p} = 2x'\Sigma(x_2 - x_1) \cdot \frac{1}{\bar{r}_2 - \bar{r}_1}$
Prop: $x_{MVP} = x_1$ ($\because$) $x_1'\Sigma(x_2 - x_1) = \frac{1}{\iota'\Sigma^{-1}\iota}\iota'(x_2 - x_1) = 0$
Prop: MVF = Hyperbola in $\bar{r} - \sigma$ plane
($\because$) $x_p = x_1 + \lambda(x_2 - x_1)$, and $\sigma_p = \sqrt{\sigma_1^2 + \lambda^2 \text{Var}(r_2 - r_1)}$ so:
$\bar{r}_p = \bar{r}_1 + \lambda(\bar{r}_2 - \bar{r}_1) = \bar{r}_1 \pm \frac{\sigma_p^2 - \sigma_1^2}{\text{Var}(r_2 - r_1)}(\bar{r}_2 - \bar{r}_1)$

Theorem $\forall$ Portf p: $\text{Cov}(r_p, r_{MVP}) = \sigma_{MVP}^2$
($\because$) New Portf α: $r_\alpha := \alpha r_p + (1 - \alpha) r_{MVP}$ so by MVP def:
$0 = \text{argmin}_\alpha \text{Var}(r_\alpha) \implies \frac{\partial}{\partial \alpha}\text{Var}(r_\alpha) = 0$ at $\alpha = 0$
$0 = \frac{\partial}{\partial \alpha}|_{\alpha=0} \left[(1 - \alpha)^2\sigma_{MVP}^2 + \alpha^2\text{Var}(r_p) + 2\alpha(1 - \alpha)\text{Cov}(r_p, r_{MVP})\right]$
$\implies 0 = -\sigma_{MVP}^2 + \text{Cov}(r_p, r_{MVP})$

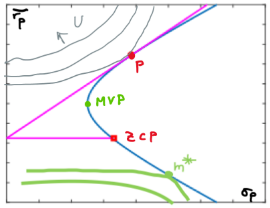
Note: Can also say that o'wise, we would get $\sigma_\alpha < \sigma_{MVP}$:
$\text{Var}(r_\alpha) = \sigma_{MVP}^2 + 2\alpha\text{Cov}(r_p - r_{MVP}, r_{MVP}) + \alpha^2\text{Var}(r_p - r_{MVP})$
$\approx \sigma_{MVP}^2 + 2\alpha\text{Cov}(r_p - r_{MVP}, r_{MVP})$ so if Cov $\gtrless 0$: take $\alpha \lessgtr 0 \downarrow \sigma_\alpha$
Prop: MVP always dominated by other MVF portfs (unless U is vertical $\to \infty$ Risk Aversion)

Definition (Zero-Covariance Portf − ZCP) Given MVF portf p:
Portf ZCP s.t. $\text{Cov}(r_{ZCP}, r_p) = 0$ for that $p \ne MVP$

Theorem (ZCP $\exists$) If MVF $p \ne MVP$, $\exists$ ZCP: $\text{Cov}(r_{ZCP}, r_p) = 0$
Note: $r_{ZCP} := r_p + \alpha(r_{MVP} - r_p)$ with $\alpha = -\frac{\sigma_p^2}{\sigma_{MVP}^2 - \sigma_p^2}$
($\because$) $\text{Cov}(r_p, r_{ZCP}) = \sigma_p^2 + \alpha\text{Cov}(r_p, r_{MVP} - r_p) = \sigma_p^2 + \alpha(\sigma_{MVP}^2 - \sigma_p^2)$

Theorem (Towards Zero-Beta CAPM) Given MVP p w/ its ZCP:
$\forall$ portf q: $\bar{r}_q - \bar{r}_{ZCP} = \beta_{qp}(\bar{r}_p - \bar{r}_{ZCP})$, where $\beta_{qp} = \frac{\text{Cov}(r_q, r_p)}{\sigma_p^2}$
($\because$) Let MVF Portf q^* with same return as q:
$r_q = r_{q^*} + r_u$ with $\mathbb{E}[r_u] = 0$, $\text{Cov}(r_u, r_p) = \text{Cov}(r_u, r_{MVP}) = 0$
$\implies \exists \alpha$ s.t. $r_{q^*} = r_{ZCP} + \alpha(r_p - r_{ZCP})$
$\implies \bar{r}_q = \bar{r}_{q^*} = \bar{r}_{ZCP} + \alpha(\bar{r}_p - \bar{r}_{ZCP})$ (so $\beta_{qp} := \alpha$) and
$\text{Cov}(r_q, r_p) = \text{Cov}(r_q - r_u, r_p) = \text{Cov}(r_{q^*}, r_p) = \alpha\sigma_p^2$

Theorem (Geometry of ZCP)



Any MVF Portf q satisfies:
$r_q = r_p + \alpha(r_p - r_{ZCP})$
$\implies \bar{r}_q = \bar{r}_p + \alpha(\bar{r}_p - \bar{r}_{ZCP})$
$\sigma_q^2 = \sigma_p^2 + 2\alpha\sigma_p^2 + \alpha^2\text{Var}(r_p - r_{ZCP})$
$\implies$ Tangent at p: Intercept=$\bar{r}_{ZCP}$
& Slope = $\frac{\bar{r}_p - \bar{r}_{ZCP}}{\sigma_p}$

Connection with the SPD:

Assumptions NA $\implies \exists$ SPD $m > 0$ (or η)
Definition m^* = projection of m on linear space of payoffs
Prop: $(m - m^*) \perp$ payoffs, so $\mathbb{E}[(m - m^*) \cdot \text{payoff}] = 0$
Prop: At $t = 0$: m^* price $P_0(m^*) = \mathbb{E}[m \cdot m^*] = \mathbb{E}[(m^*)^2]$
($\because$) $\mathbb{E}[mm^*] = \mathbb{E}[(m^* + (m - m^*))m^*] = \mathbb{E}[(m^*)^2] + \mathbb{E}[(m - m^*) \cdot m^*]$
Prop: Return on portf m^*: $1 + r^* := \frac{m^*}{P_0(m^*)} = \frac{m^*}{\mathbb{E}[(m^*)^2]}$

Proposition (SDF Prices All Assets) $\mathbb{E}[m^* \cdot \text{payoff}] = 0$
Note: Also, $\mathbb{E}[r^* \cdot \text{payoff}] = 0$ (as $r^* \propto m^* \propto m$)
($\because$) TBD

Proposition (Efficiency of m^*) Portf m^* is on MVF
($\because$) Decompose $r^* = \frac{m^*}{\mathbb{E}[(m^*)^2]}$:
$r^* = r_p + r_u$ with $\mathbb{E}[r_u] = 0$, $\text{Cov}(r_u, r_p) = \text{Cov}(r_u, r_{MVP}) = 0$
Notice: $\mathbb{E}[r_p r_u] = \text{Cov}(r_p, r_u) + \mathbb{E}[r_p]\mathbb{E}[r_u] = 0 + \mathbb{E}[r_p] \cdot 0 = 0$
$\implies \mathbb{E}[r^* r_u] = \mathbb{E}[r_p r_u] + \mathbb{E}[r_u^2] = 0 + \sigma_u^2$
Also: $\mathbb{E}[r^* r_u] = 0$ as $r^* \propto m^* \propto m$ and m prices all portfs: $\mathbb{E}[r_u m] = 0$
$\implies \sigma_u^2 = 0 \implies r_u \equiv 0 \implies r^* = r_p$ so m^* is on the MVF

Proposition (MVF Location of m^*)
m^* minimizes $\mathbb{E}\left[r_p^2\right] = \sigma_p^2 + \bar{r}_p^2$ among all MVF portfs
($\because$) For any MVF portf p: $r_p = r^* + \alpha(r_p - r_{ZCP})$
$\implies \mathbb{E}\left[r_p^2\right] = \mathbb{E}[(r^*)^2] + \alpha^2\mathbb{E}[(r_p - r_{ZCP})^2] + 2\alpha\mathbb{E}[r^*(r_p - r_{ZCP})]$
where $\mathbb{E}[r^*(r_p - r_{ZCP})] = 0$ as SDF prices all payoffs
$\implies \mathbb{E}\left[r_p^2\right] \ge \mathbb{E}[(r^*)^2]$ for all $p \in$ MVF

Riskless Asset:

Assumptions N risky assets, 1 RF asset (rate r^f)
- **Excess return** on risky assets: $r^e = r - r^f\iota$
- **Portf weights** on risky assets: x, on RF asset: $1 - \iota' x$
- **Terminal Wealth:** $\tilde{w} = w\left[(1 + r^f) + x'r^e\right]$

Definition (MVF Portfolio) Expected Excess Return $\bar{r}_p^e$
$x_p = \text{argmin}_x x'\Sigma x$ s.t. $\bar{r}^e x = \bar{r}_p^e$ **Note:** $\bar{r}^e \in \mathbb{R}^N$ but $\bar{r}_p^e \in \mathbb{R}$

Theorem (MVF Solution) $\mathcal{L} = \frac{1}{2}x'\Sigma x + \lambda(\bar{r}_p^e - x'\bar{r}^e)$

FOC: $x'\Sigma = \lambda \bar{r}^e$ with $\bar{r}_p^e = x'\bar{r}^e x \implies x_p = \frac{\bar{r}_p^e}{\bar{r}^{e'}\Sigma^{-1}\bar{r}^e}\Sigma^{-1}\bar{r}^e$

Vary $\bar{r}_p^e$ to get the MVF in presence of RF asset

Note: $\bar{r}_p^e = 0 \implies x_p = 0$ so all in RF

$\bar{r}_p^e = \frac{\bar{r}^{e'}\Sigma^{-1}\bar{r}^e}{\bar{r}^{e'}\Sigma^{-1}\iota} \implies \iota' x_p = 1$ so all in risky $\equiv$ **Tangency Portf**

In general: total weight in risky asset $a_p := \iota' x_p = \frac{\bar{r}^{e'}\Sigma^{-1}\iota}{\bar{r}^{e'}\Sigma^{-1}\bar{r}^e}\bar{r}_p^e$

Definition (Tangency Portf) All in risky asset:

$x_T := \frac{1}{\bar{r}^{e'}\Sigma^{-1}\iota}\Sigma^{-1}\bar{r}^e$ **Prop:** For all $p \in$ MVF: $a_p = \frac{\bar{r}_p^e}{\bar{r}_T^e}$, $x_p = a_p x_T$

Also we get a line: $r_p = r^f + w(r_T - r^f)$

Any MVF portf = mix of Tangent portf & RF asset

Definition (Sharpe Ratio) Given Portf x: $SR := \frac{x'\bar{r}^e}{\sqrt{x'\Sigma x}}$

Prop: Among all portfs of risky assets only: Tngt portf has max SR

$(\because)$ FOC to max SR: $x = \frac{\sqrt{x'\Sigma x}}{SR}\Sigma^{-1}\bar{r}^e \implies \iota' x = 1 \implies x =$ Tngt

Theorem Let $p \in$ MVF, then for all portf q:

$\bar{r}_q - r^f = \beta_{qp}(\bar{r}_p - r^f), \quad \beta_{qp} := \frac{\text{Cov}(r_p, r_q)}{\sigma_p^2}$

$(\because)$ $x_p = \frac{\bar{r}_p^e}{\bar{r}^{e'}\Sigma^{-1}\bar{r}^e}\Sigma^{-1}\bar{r}^e$

$\implies \text{Cov}(r_p, r_q) = \frac{\bar{r}_p^e}{\bar{r}^{e'}\Sigma^{-1}\bar{r}^e}\bar{r}^{e'}\Sigma^{-1}\Sigma x_q = \frac{\bar{r}_p^e}{\bar{r}^{e'}\Sigma^{-1}\bar{r}^e}\bar{r}_q^e$

If $p = q$: $\sigma_p^2 = \frac{(\bar{r}_p^e)^2}{\bar{r}^{e'}\Sigma^{-1}\bar{r}^e} \implies \text{Cov}(r_p, r_q) = \frac{\bar{r}_q^e}{\bar{r}_p^e}\sigma_p^2 = \beta_{qp}$

Long - Short:

$r_h = r_q - r_f$

perturbed portf: $r_\alpha := r_p + \alpha r_h \Rightarrow \begin{cases} \bar{r}_\alpha = \bar{r}_p + \alpha(\bar{r}_q - r_f) \\ \sigma_\alpha^2 = \sigma_p^2 + 2\alpha \text{Cov}(r_q - r_f, r_p) + \alpha^2\sigma_h^2 \end{cases}$ (α small)

$\left.\frac{d\sigma_\alpha}{d\alpha}\right|_{\alpha=0} = \frac{1}{2}\frac{2\text{Cov}(r_q - r_f, r_p)}{\sigma_p} = \frac{\text{Cov}(r_q - r_f, r_p)}{\sigma_p}$ by optimality: I cannot move outside of the line (up/down)

$\Rightarrow \frac{\bar{r}_q - r_f}{\text{Cov}(r_q - r_f, r_p)} \cdot \sigma_p \overset{!}{=} SR = \frac{\bar{r}_p - r_f}{\sigma_p} \Rightarrow \beta = \frac{\text{Cov}(r_q - r_f, r_p)}{\sigma_p^2}$

Note:

- Under mean variance prefs: 2-fund separation holds.
- The optimal portfolios of all agents have a very simple structure.

Portfolio efficiency: basic intuition

- Recall the first order condition for the tangency portfolio:

  $$0 = x^\top\Sigma - \lambda\bar{\pi}$$

- Rearranging, this implies that

  $$\pi_j = \bar{r}_j - r_F = \frac{1}{\lambda}\mathbb{C}(r_j, r_p) = \frac{1}{\lambda}\frac{\partial(1/2x^\top\Sigma x)}{\partial x_j}$$

  - LHS is the marginal benefit (increase in expected return) associated with borrowing at r_F to increase the weight in asset j
  - RHS is the corresponding marginal cost (increase in variance). λ is the shadow cost (in terms of variance) of a marginal increase in expected returns.
  - An efficient portfolio will equate the two across all risky assets.
- If an asset earns a high risk premium, it must be the case that a marginal increase in its weight would generate a larger increase in portfolio variance (i.e., it has a higher covariance with the tangency portfolio) relative to another asset with a lower risk premium.

In Practice

Consumption Choice Problem 1:

Setup: M=2 states, N=2 assets (1 RF + 1 risky)

t=1 returns: $R = \begin{bmatrix} 1 & u \\ 1 & d \end{bmatrix}$, probs: π_u, π_d.

Agent: initial wealth w_0, final $w_1 = 0$.

Portf Weights (RF/Risky asset): $\alpha = [\alpha_1, \alpha_2]'$

Max Prob: $\max_{c_0, c_1, \alpha} \log c_0 + \beta\mathbb{E}[\log c_1]$

2 Budget Constraints: $c_1 = (w_0 - c_0)R\alpha$

(a) Find optimal $\alpha = \alpha(w_0, c_0, c_1)$:

Complete Market: $\exists R^{-1} \implies \alpha = \frac{1}{w_0 - c_0}R^{-1} \cdot c_1$

(b) Rewrite Constraints Using Only (w_0, c_0, c_1):

$\phi = P'R^{-1}$ with $P = [1, 1]'$ so $\eta_u = \frac{\phi_u}{\pi_u}, \eta_d = \frac{\phi_d}{\pi_d}$

$\implies$ Constraint: $w_0 := c_0 + \mathbb{E}[\eta c_1] = c_0 + \pi_u\eta_u c_{1,u} + \pi_d\eta_d c_{1,d}$

(c) Optimize over (c_0, c_1) + Find $c_0(\lambda), c_1(\lambda)$:

$\max_{c_0, c_1} \log c_0 + \beta\mathbb{E}[\log c_1]$ s.t. $w_0 = c_0 + \pi_u\eta_u c_{1,u} + \pi_d\eta_d c_{1,d}$

$\mathcal{L} = \log c_0 + \beta\mathbb{E}[\log c_1] - \lambda(c_0 + \pi_u\eta_u c_{1,u} + \pi_d\eta_d c_{1,d} - w_0)$

$\overset{c_0}{\implies} \lambda = 1/c_0$; $\overset{c_{1,s}}{\implies} \lambda\pi_s\eta_s = \beta\pi_s/c_{1,s}$ with $s = u, d$

$\implies c_0 = \frac{1}{\lambda} \quad c_{1,u} = \frac{\beta}{\lambda\eta_u} = \frac{c_0\beta}{\eta_u} \quad c_{1,d} = \frac{\beta}{\lambda\eta_d} = \frac{c_0\beta}{\eta_d}$

(d) Plug λ in constraint + Get $\lambda = \lambda(w_0)$:

$w_0 = c_0 + \pi_u\eta_u c_{1,u} + \pi_d\eta_d c_{1,d} = \frac{1}{\lambda} + \frac{\pi_u\beta}{\lambda} + \frac{\pi_d\beta}{\lambda}$

$\implies \lambda = \frac{1+\beta}{w_0}$

(e) Get $c_0(w_0), c_1(w_0)$:

$c_0 = \frac{w_0}{1+\beta} \quad c_{1,u} = \frac{\beta}{1+\beta}\frac{w_0}{\eta_u} \quad c_{1,d} = \frac{\beta}{1+\beta}\frac{w_0}{\eta_d}$

Consumption Choice Problem 2:

Setup: 1 RF (return r^f) + 1 Risky asset:

$r = \bar{r} + \sigma\varepsilon \quad \varepsilon \sim N(0, 1) \quad (\bar{r} > r^f)$

Agent: $e_0 > 0$ and $e_1 = h\varepsilon$ $(h > 0)$

Maximize: $\max_{c_0, c_1} -e^{-\alpha c_0} - \rho\mathbb{E}\left[e^{-\alpha c_1}\right]$, $\alpha > 0$ cst

(a) Invest a in risky asset:

Write $t = 1$ consumption $c_1 = c_1(e_0, e_1, c_0, a, r^f, r)$

$c_1 = e_1 + w(1 + r^f) + a(r - r^f) = e_1 + (e_0 - c_0)(1 + r^f) + a(r - r^f)$

(b) Write optimal portf choice problem:

$\max_{c_0, a} -e^{-\alpha c_0} - \rho\mathbb{E}\left[e^{-\alpha\left(e_1 + (e_0 - c_0)(1 + r^f) + a(r - r^f)\right)}\right]$

(c) Write FOC: $\frac{d}{da}$ and $\frac{d}{dc_0}$

(d) Solve FOC for Opt Portf Choice problem: Get a, c_0

(e) How does h influences c_0 and a?

$c_0 \searrow$ in h. Higher uncertainty about $e_1 \implies$ lower certainty equivalent of this payoff. When h is high: agent feels poorer $\Rightarrow$ wants to consume less.

$a \searrow$ in h. Higher uncertainty about $e_1 \implies$ less willingness to invest in the risky asset (adds risk to c_1).

Consumption Choice Problem 3:

Setup: 1 RF (asset 0) +N risky assets: $n = 1, .., N$

Returns: $R_0 = R^f = 1$ and $D_n = \bar{D} + \varepsilon_n$ $(n = 1..N)$, $\varepsilon_n \overset{\text{iid}}{\sim} N(0, \sigma)$

Prices: $P_0 = 1$, $P_n = P$ for all n

Agent: CARA $u(w) = -e^{-aw}$, $a > 0$

Endowment: 1 share of each asset n, 0 shares of RF.

Portfolio holdings of risky assets: $\theta = [\theta_1, \ldots, \theta_N]' \in \mathbb{R}^N$

(a) Write agent's wealth $\tilde{w}$ at $t = 1$:

$$\tilde{w} = \sum_{n=1}^N \underbrace{\theta_n D_n}_{\substack{\text{Asset } n \\ \text{Payoff}}} + \frac{1}{P_0}\Big(\sum_{n=1}^N \underbrace{P_n}_{\substack{\text{t=0 wealth} \\ \text{(endowment)}}} - \sum_{n=1}^N \underbrace{\theta_n P_n}_{\substack{\text{wealth invested} \\ \text{in risky assets}}}\Big)$$

(b) Write Optimal Portf Choice Pb:

$\max_\theta \mathbb{E}\left[-e^{-a\tilde{w}}\right]$ s.t. $\tilde{w} = \sum_{n=1}^N \theta_n D_n + \sum_{n=1}^N (1 - \theta_n)P_n$

$D_n \overset{\text{iid}}{\sim} N(\bar{D}, \sigma^2)$

$\implies \tilde{w} \sim N\left(\sum_{n=1}^N \theta_n\bar{D}_n + \sum_{n=1}^N (1 - \theta_n)P_n, \sigma^2\sum_{n=1}^N \theta_n^2\right)$

$\Rightarrow -a\tilde{w} \sim N\left(-a\sum_{n=1}^N \theta_n\bar{D}_n - a\sum_{n=1}^N (1 - \theta_n)P_n, a^2\sigma^2\sum_{n=1}^N \theta_n^2\right)$

$\mathbb{E}\left[-e^{-a\tilde{w}}\right] = -\exp\left(-a\sum_{n=1}^N \theta_n\bar{D}_n - a\sum_{n=1}^N (1 - \theta_n)P_n + \frac{a^2}{2}\sigma^2\sum_{n=1}^N \theta_n^2\right.$

$$\implies \max_\theta \sum_{n=1}^N \theta_n\bar{D}_n + \sum_{n=1}^N (1 - \theta_n)P_n - \frac{a}{2}\sigma^2\sum_{n=1}^N \theta_n^2$$

(c) Solve Optimal Portf Pb: FOC w.r.t. θ_n

$\bar{D} - P_n - a\sigma^2\theta_n = 0 \implies \theta_n = \frac{\bar{D} - P_n}{a\sigma^2}$

(d) Show: for different values of RA a, 2-Fund Separation Holds:

Initial Wealth of Agents: $w_0 = \sum_{n=1}^N P_n$

$\implies$ agents invest optimally fractions

$\frac{\theta_n P_n}{w_0} = \frac{1}{a}\frac{(\bar{D} - P_n)P_n}{w_0\sigma^2}$ of wealth in asset n

and $1 - \frac{1}{a}\frac{(\bar{D} - P_n)P_n}{w_0\sigma^2}$ in RF asset

$\implies$ agents hold lin comb of RF asset

and risky portf $x_M = [x_1, \ldots, x_N]'$ $\left(x_n = \frac{(\bar{D} - P_n)P_n}{w_0\sigma^2}\right)$

Depending on RA: Hold $\frac{1}{a}X_M$ in risky & $1 - \frac{1}{a}X_M'\iota$ in RF

(e) If agent = only agent in market

Find Equilibrium Risky Prices P_n

Market Clearing: $1 \overset{!}{=} \theta_n = \frac{\bar{D} - P_n}{a\sigma^2} \implies P_n = \bar{D} - a\sigma^2$

(f) Find Risk Premium on Risky Assets + $N \to \infty$ Limit:

$\pi_n = \bar{R}_n - R^f = \mathbb{E}[R_n] - 1 = \frac{\mathbb{E}[D_n]}{P_n} - 1 = \frac{a\sigma^2}{\bar{D} - a\sigma^2}$ indep of N

(g) Does APT Hold in this Market when $N \to \infty$?

$\exists$ Asymptotic Arbitrage in this Market: (so APT can't hold)

Seq of arb portfs: $\theta_0^N = -1$, $\theta_n^N = \frac{1}{N}$ $\forall n$

$\implies \mathbb{E}\left[r_{\theta^N}\right] = \sum_{n=1}^N \frac{1}{N}\bar{r}_n - 1 = \bar{r}_1 - 1 = \pi_1 = \frac{a\sigma^2}{\bar{D} - a\sigma^2} > 0$

While $\text{Var}\left(r_{\theta^N}\right) = \sum_{n=1}^N \frac{1}{N^2}\text{Var}(\varepsilon_n) = \frac{\sigma^2}{N} \to 0$

Static Equilibrium Models of Asset Pricing

Market Equilibrium

Invest - Transfer money:

- Deposit accounts
- Mortgages

The Capital Asset Pricing Model (CAPM)

The Consumption-based CAPM (C-CAPM)

Asymmetric Information

Financial Markets: Grossman-Stiglitz Model

Invest - Transfer money:

- Deposit accounts
- Mortgages

No-Trade Theorem

Rational Expectation/Market Efficiency

Market Microstructure: Kyle & Glosten-Milgrom Models

Dynamic Modelling

Dynamic State-Space Framework, FTAP

Invest - Transfer money:

- Deposit accounts
- Mortgages

Arbitrage Asset Pricing (Dynamic)

Dynamic Portfolio Choices

Dynamic Equilibrium Models: Complete Markets, CCAPM

Dynamic Equilibrium Models: Incomplete Markets