# Fundamental Theory of Asset Pricing

$X \sim N(\mu,\sigma^2)$: $M_X(t) = \mathbb{E}\left[e^{-tX}\right] = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$

$M_X(t) = \mathbb{E}\left[e^{-\sum_{i=1}^n t_i X_i}\right] = \exp\left(t'\mu + \frac12 t'\Sigma t\right)$, with $t \in \mathbb{R}^n$.

## Arrow-Debreu State-Space Framework

**$t = 1$ State Space:** $\Omega = \{\omega_1,\dots,\omega_M\} \implies$ **M states**
**K Agents** $k=1,2,\dots,K$ **N Assets** $n=1,2,\dots,N$
**Endowment:** $\mathbf{e^k} := [e_0^k, \mathbf{e_1^k}']' = [e_0^k, e_{11}^k, \dots, e_{1M}^k]' \in \mathbb{R}^{1+M}$
**Consump Plan:** $\mathbf{c^k} := [c_0^k, \mathbf{c_1^k}']' = [c_0^k, c_{11}^k,\dots,c_{1M}^k]' \in \mathbb{R}^{1+M}$
**Consump. Set:** $C := \{\mathbf{c^k} : \text{feas.}\} \subseteq \mathbb{R}^{1+M}$; Usual: $C = \mathbb{R}_+^{1+M}$
**Prop:** $C$ is a closed + convex subset of $\mathbb{R}^{1+M}$.
**Budget Set:** Set of consumption plans given by purchasing $\theta$:
$B(e,\{D_1,P\}) := \{c \geq 0 : \theta \in \mathbb{R}^N, c_0^k = e_0 - P'\theta, c_1^k = e_1 + D_1\theta\}$
**Complete, No redund:** $\text{rank}(D_1) = M = N \implies \theta = D^{-1}(c_1 - e_1)$
$\implies B(e,\{D_1,P\}) = \{c_1 \in \mathbb{R}_+^M : P'D_1^{-1}(c_1-e_1) \leq e_0\}$
**Payoff Vector:** $D_{1n} = [D_{11},\dots,D_{1M}]' \in \mathbb{R}^M$, $D_{1\omega}$ at $t=1$ state $\omega$.
**Market Structure:** $D_1 = [D_{11},\dots,D_{1N}] = [D_{1\omega n}]_{M\times N} \in \mathbb{R}^{M\times N}$
**Price:** $P_n \in \mathbb{R}$ : price of security $n$ at $t=0$.
**Price Vector:** $P = [P_1,\dots,P_N]' \in \mathbb{R}^N$ at $t=0$.
**Portfolio:** $\theta = [\theta_1,\dots,\theta_N] \in \mathbb{R}^N \Rightarrow \text{cost}_{t=0} = -P'\theta$, $\text{payoff}_{t=1} = D_1\theta$.
**$B$ Matrix:** $B := [-P', D]' \in \mathbb{R}^{(M+1)\times N}$.
**Prop:** $c = e + B'\theta = [e_0 - P'\theta, [e_{1m} + \sum_{n=1}^N D_{n1m}\theta_n]'_{M\times 1}]$
**Market Clearing:** $\sum_{k=1}^K \theta^k(P,e^k) = 0$ i.e. $\sum_{k=1}^K c^k = \sum_{k=1}^K e^k$.

## Arbitrage

**Redundant Security:** $\exists \theta_{\setminus n}$ s.t. $D_{\setminus n}\theta_{\setminus n} = D_n$
**Prop:** $\text{rank}(D_1) < N$=redundant; $\text{rank}(D_1) < M$=Incompl
$(Im(D_1) \subsetneq \mathbb{R}^M)$; $\text{rank}(D_1) = N = M$= Compl + no redund
**Payoff Space:**
$C_1(D_1) := \{c_1 = D_1\theta \in \mathbb{R}^M : \theta \in \mathbb{R}^N\} = \text{span}(D_1,\dots,D_N) \subseteq \mathbb{R}^M$
**Complete Market:** $\text{rank}(D) = M$ (need $N = M$)
**AD Securities:** $D^{AD} := \mathbb{I}_{N\times N}$ (complete market):
$\to$ construct using portfolio $\theta = D^{-1}$ $(\because)$ $D\theta = \mathbb{I} = D^{AD}$
**State Prices:** $\phi = (D^{-1})'P$, $P' = \Phi' D$; $\phi_\omega$ = price of $e_\omega$ at $t=0$.
**Arbitrage Existence:** $\iff \exists\theta\in\mathbb{R}^N$ s.t. $B\theta > 0 \iff \exists\phi \gg 0$
ex: **CIP Formula** Forward contract price: $F = S_{€\to\$} \times \frac{1+r_\$}{1+r_€}$

## Fundamental Thm of Asset Pricing (FTAP)

**Valuation Op:** $V$ positive Linear Operator
**Theorem (FTAP)** NA $\iff \exists\phi \gg 0$ s.t. $P' = \phi' D$ $(P = D'\phi)$
**Note:** If $\text{rank}(D) = N$, then $D'D$ full rank: $P' = \theta' D' D$.
$\implies \theta^* = (D'D)^{-1}P$, $\phi^* = D(D'D)^{-1}P$ and $\eta^* = \phi^*\text{Diag}(p)^{-1}$
**Note:** Redund asset $\implies \phi$ not unique. If $\nexists\phi\gg 0 \implies \exists$ arb
**Example: (Incomplete Market)** $D = (1,2,3)'$, $P = 1$:
$\implies \Phi = \{\phi \gg 0 : \phi' D = 1\} = \{\phi\gg 0 : \phi_1 + 2\phi_2 + 3\phi_3 = 1\}$.
Price $D_2 = (2,2,2)' : P_2$
**Example: (Incomplete Market II)** Find price $P_b$ of new security with payoff b s.t. $b \notin \text{span}(D_1)$ (i.e., not redundant):
$P_b = \{\phi b : \phi \gg 0, P' = \phi' D_1\}$ i.e., $\inf_{\phi\in\Phi\gg 0} \phi' b < P_b < \sup_{\phi\in\Phi\gg0}\phi' b$
If Security b is redundant: $P_b = \{P\theta : \theta\in\mathbb{R}^n, D_1\theta = b\} = \phi' b$.

## State-Price Density/Risk Neutral Measure

**Risk-Neutral Pricing:**
1. $P' = \phi' D \implies$ get state prices $\phi$.
2. Get RF rate: $1 + r^f = \frac{1}{\sum_\omega \phi_\omega}$.
3. Construct $\mathbb{Q} = \left\{q_\omega = \frac{\phi_\omega}{\sum_{\omega'}\phi_{\omega'}} = (1+r^f)\phi_\omega\right\}$.
4. Price any asset with payoff vector $D_n$:
$P_n = \frac{\mathbb{E}^{\mathbb{Q}}[D_n]}{1+r^f} = \frac{1}{1+r^f}\sum_\omega q_\omega D_{n\omega}$.
5. Get **Expected Return:** $1 + \bar r_n = (1+r^f)\frac{\mathbb{E}^{\mathbb{P}}[D_n]}{\mathbb{E}^{\mathbb{Q}}[D_n]}$.

**SDF/SPD:** $P_n = \phi' D_n = \sum\phi_\omega D_{n\omega} = \sum p_\omega \frac{\phi_\omega}{p_\omega} D_{n\omega}$
**Proposition** $(\mathbb{P}\sim\mathbb{Q})$ $\mathbb{E}^{\mathbb{P}}[X] = \frac{1}{1+rf}\mathbb{E}^{\mathbb{Q}}\left[\frac{X}{\eta}\right]$, & $\phi_\omega = \frac{q_\omega}{1+r^f} = p_\omega\eta_\omega$

**SPD/SDF Pricing: you must know $\mathbb{P} = \{p_\omega\}$:**
1. $P' = \phi' D \implies$ get state prices $\phi$.
2. Get SPD/SDF: $\eta = \frac{\phi_\omega}{p_\omega}$.
3. Price any asset with payoff vector $D_n$:
$P_n = \mathbb{E}^{\mathbb{P}}[\eta D_n] = \sum_\omega p_\omega\eta_\omega D_{n\omega}$.
4. Get **Expected Return:** $1+\bar r_n = \frac{\mathbb{E}^{\mathbb{P}}[D_n]}{P_n} = \frac{\mathbb{E}^{\mathbb{P}}[D_n]}{\mathbb{E}^{\mathbb{P}}[\eta D_n]}$.
5. Get RF rate: $1 + r^f = \frac{1}{\mathbb{E}^{\mathbb{P}}[\eta]}$.

**Discounted Cash Flow (DCF)/Present Value (PV) Formula:**
**Discount Rate/Expected Rate of Return:**
$1+\bar r_n = \frac{\mathbb{E}^{\mathbb{P}}[D_n]}{P_n} = \frac{\mathbb{E}^{\mathbb{P}}[D_n]}{\mathbb{E}^{\mathbb{P}}[\eta D_n]} = (1+r^f)\frac{\mathbb{E}^{\mathbb{P}}[D_n]}{\mathbb{E}^{\mathbb{Q}}[D_n]} = \frac{\mathbb{E}^{\mathbb{Q}}[D_n/\eta]}{\mathbb{E}^{\mathbb{Q}}[D_n]}$.
**Gross Return:** $R_{t+1} = \frac{\text{Payoff(t+1)}}{\text{Payoff(t)}} = \frac{X_{t+1}}{P_t} = \frac{D_t + P_t}{P_t}$
**Net Ret:** $r_t = 1 + R_t$. Random $\tilde r_n : r_{n\omega} = \frac{D_{n\omega}}{P_n} - 1$
**Note:** $\tilde r_{1\omega} = r^f$, $\mathbb{E}^{\mathbb{P}}[\tilde r_n] = \bar r_n$, and $1+\tilde r_n = \frac{D_n}{P_n}$
**Excess Ret:** $= r_t - r^f$. **Risk Prem:** $\pi_t = \mathbb{E}^{\mathbb{P}}\left[\tilde r_n - r^f\right] = \bar r_n - r^f$.
**Sharpe:** $SR_t = \frac{\mathbb{E}[r_t - r^f]}{\text{Std}(r_t - r^f)} = \frac{\mathbb{E}[\text{return}]}{\text{Unit Risk}}$.
**Prop:** (DCF/PV) $P_n = \frac{\mathbb{E}^{\mathbb{P}}[D_n]}{1+\bar r_n} = \frac{\sum_\omega p_\omega D_{n\omega}}{1+\bar r_n} = \phi' D_n$
$P_1 = \sum_\omega\phi_\omega = \frac{1}{1+r^f}$ and $\bar r_1 = \frac{\mathbb{E}^{\mathbb{P}}[D_1]}{P_1} - 1 = \frac{1}{P_1} - 1 = r_f^f$
**Prop:** Risk Prem $\mathbb{E}^{\mathbb{P}}[1+\tilde r_n] = (1+r^f)\left(1 - \text{Cov}^{\mathbb{P}}(\eta, 1+\tilde r_n)\right)$
$\implies \pi_n = \mathbb{E}^{\mathbb{P}}\left[\tilde r_n - r^f\right] = -(1+r^f)\text{Cov}^{\mathbb{P}}\left(\eta, \tilde r_n - r^f\right)$
and $\mathbb{E}^{\mathbb{Q}}[\tilde r_n] = r^f \;\forall n \longrightarrow \mathbb{E}^{\mathbb{Q}}\left[\tilde r_n - r^f\right] = 0\;\forall n$
**Note:** $\rho(\eta, D_n) = 0 \implies \bar r_n = r^f$ (but $\tilde r_n \neq r^f$) and $P_n = \frac{\mathbb{E}^{\mathbb{P}}[D_n]}{1+r^f}$
**Prop:** (Hansen-Jagannathan) $SR_n := \frac{\mathbb{E}^{\mathbb{P}}[\tilde r_n - r^f]}{\sqrt{\text{Var}^{\mathbb{P}}(\tilde r_n - r^f)}} \leq \frac{\sqrt{\text{Var}^{\mathbb{P}}(\eta)}}{\mathbb{E}^{\mathbb{P}}[\eta]}$

## FTAP: Corporate Finance

$y_0$: investment into the production opportunity at $t=0$
$y_{1\omega} = y_\omega(y_0)$, $\forall\omega\in\Omega$: prod output at $t=1$ $y_\omega(0) = 0, y'_\omega \geq 0, y''_\omega < 0$
**Investment NPV** $v = \phi' y - y_0 = \sum_\omega \phi_\omega y_\omega(y_0) - y_0$
**Agent's $t=0$ Wealth** $w = e_0 - y_0 + \phi'(e_1 + y_1) = e_0 + \phi' e_1 + v$
**Agent's Opt Pb:** $\max_{y_0,c_0,c_1} u([c_0, c_1']')$ s.t. $w = c_0 + \phi' c_1$
**Agent's Consumption:** $c_0 = e_0 - y_0$, $c_1 = e_1 + y_1$
**Corporate Investment Decisions:** $j = 1..F$ firms: $y_j(y_{j0}) \in \mathbb{R}^M$
$s_{kj}$ = share of firm $j$ owned by agent $k$: $\sum_k s_{kj} = 1, \forall j$
Firm's investment NPV at $t=0$: $v_j = \phi' y_j(y_{j0}) - y_{j0}$, $\forall k$ (MAX IT)
Agent $k$'s wealth: $w_k = e_{k0} + \phi' e_{k1} + \sum_j s_{kj} v_j$
**Prop:** $y_1 = d_1 + e_1$, $D = \phi' d_1$, $E = \phi' e_1$ and $V = D + E = \phi' y_1$.
NPV(Equity)$= \phi'(y_1 - d_1) - e_0 = \phi' y_1 - (d_0 + e_0) = \phi' y_1 - y_0$
**Example: (Labor vs Wage)** Firm: hire $L$ labor at wage $W$
$\implies$ produce output $Y(L) = AL^\alpha$ $(\alpha < 1)$, $\log A \sim N(\bar A, \sigma_A^2)$
Can borrow at $r^f$, and Log-SDF: $\log M = \delta + \varepsilon$, $\varepsilon \sim N(0,\sigma_\varepsilon^2)$
Firm's pb: $\max_L -WL + \mathbb{E}[MAL^\alpha] \implies L = (\alpha\mathbb{E}[MA]/W)^{1/(1-\alpha)}$

## FTAP: Fundamental Value of a Stocks

**CFs:** $X_t = D_t + P_t$ (div + share price) **Ret:** $R_t = \frac{X_t}{P_t} = \frac{D_t + P_t}{P_t}$.
**(PV)** $P_t = \mathbb{E}_t^{\mathbb{P}}[\eta_{t+1}(P_{t+1} + D_{t+1})] = \mathbb{E}_t^{\mathbb{Q}}\left[\sum_{k=1}^\infty \frac{D_{t+k}}{(1+r^f)^k}\right]$
$\implies P_t = \mathbb{E}_t^{\mathbb{P}}\left[\sum_{k=1}^\infty \eta_{t:t+k} D_{t+k}\right]$ with $\eta_{t:t+k} := \prod_{j=1}^k \eta_{t+j}$
$k^{th}$ **Period Ret** $\mathbb{E}_t^{\mathbb{P}}\left[R_{t:t+k}^{(k)}\right] = \frac{\mathbb{E}_t^{\mathbb{P}}[D_{t+k}]}{\mathbb{E}_t^{\mathbb{P}}[\eta_{t:t+k}D_{t+k}]} = \frac{\mathbb{E}_t[\text{Payoff}]}{\text{Price}}$
$\implies P_t^{(k)} = \mathbb{E}_t^{\mathbb{P}}[\eta_{t:t+k}D_{t+k}] = \mathbb{E}_t^{\mathbb{P}}[D_{t+k}]/\mathbb{E}_t^{\mathbb{P}}\left[R_{t:t+k}^{(k)}\right]$
**(DCF)** $P_t = \mathbb{E}_t^{\mathbb{P}}\left[\sum_{k=1}^\infty \eta_{t:t+k}D_{t+k}\right] = \sum_{k=1}^\infty \mathbb{E}_t^{\mathbb{P}}[D_{t+k}]/\mathbb{E}_t^{\mathbb{P}}\left[R_{t:t+k}^{(k)}\right]$

## FTAP: Fixed Income Securities

$P_{N,t}$ = price of $N$-period bond at time $t$ that pays $FV = 1\$$ at $t+N$.
1 period: $N=1$, state $s_t = j$: $P_{1,t} = P_1(j) = \mathbb{E}_t[\eta_{t+1}]$
$N$: $\mathbf{P_{N,t}} = \mathbb{E}_t[\eta_{t+1}P_{N-1,t+1}] = \mathbb{E}_t[\eta_{t+1}\times\cdots\times\eta_{t+N}] = \mathbb{E}_t[\eta_{t:t+N}]$

**Yield Curve (YC)/ Term Structure of Interest Rates:**
**YTM:** $Y_{N,t} = \left[\frac{1}{P_{N,t}}\right]^{1/N} = [\text{payoff/price}]^{1/N} \longrightarrow P_{N,t} = \left[\frac{1}{Y_{N,t}}\right]^N$
**Log-Framework** $p_{N,t} = \log P_{N,t}$, $y_{N,t} = \log Y_{N,t}$ $p_{N,t} = -N\cdot y_{N,t}$
**Proposition (YC Recipe)** $P_{N,t} = \mathbb{E}_t[\eta_{t:t+N}]$: YC$\Leftrightarrow$moments of SDF
(1) Define State Variables: $x_t$ (data $\Rightarrow$ need at least 3).
(2) Assume SDF $\eta_t = \eta(x_t)$ or Log SDF $m_t = \log\eta_t$.
(3) Give law of motion for $x_t$ under $\mathbb{P}$ (use $\pi(x_{t+1}, x_t)$)
**OR** Give law of motion for $x_t$ under $\mathbb{Q}$ (use $r^f$)
(4) Sol°: Iterate on pricing eq. $P_N(x_t) = \mathbb{E}_t[\eta(x_{t+1})P_{N-1}(x_{t+1})|x_t]$
(5) Guess $p_{N,t} = \log(P_{N,t}) \sim$ affine in $x_t$ & find coeffs

**Example: (Vasicek model, 1977)**
(1) **One State Variable:** $x_t$
(2) **Assume Log SDF:** $m_{t+1} = \log(\eta_{t+1}) = -x_t - \frac12\left(\frac{\lambda}{\sigma}\right)^2 - \frac{\lambda}{\sigma}\varepsilon_{t+1}$.
(3) **AR(1)** under $\mathbb{P}$: $x_{t+1} = \mu + \phi x_t + \sigma\varepsilon_{t+1}$; with $\phi < 1$, $\varepsilon_t \overset{\text{iid}}{\sim} N(0,1)$
(4) **Sol°:** Iterate on $p_{N,t} = \log\mathbb{E}_t[\exp(m_{t+1} + p_{N-1,t+1})|x_t]$
Use: $(x_{t+1}|\mathcal{F}_t) \sim N(\mu+\phi x_t, \sigma)$ and $(m_{t+1}|\mathcal{F}_t) \sim N\left(-x - \frac12\left(\frac{\lambda}{\sigma}\right)^2, \frac{\lambda}{\sigma}\right)$
$p_{1,t} = \log\mathbb{E}_t[\exp(m_{t+1})] = -x_t - \frac12\left(\frac\lambda\sigma\right)^2 + \frac12\left(\frac\lambda\sigma\right)^2 = 0 - 1\cdot x_t$
$\implies$ **Short Rate:** $x_t = y_{1t} = \log(1+r^f) \longrightarrow$ **mean-reverting** AR(1)
$p_{2,t} = \log\mathbb{E}_t[\exp(m_{t+1} + p_{1,t+1})] = \log\mathbb{E}_t[\exp(m_{t+1} - x_{t+1})]$
$= -(1+\phi)x_t + \left[-\frac12\left(\frac\lambda\sigma\right)^2 + \mu + \frac12\left(\frac\lambda\sigma + \sigma\right)^2\right] = A_2 + B_2\cdot x_t$
(5) **Guess** $p_{n,t} = A_n + B_n\cdot x_t$:
$p_{n,t} = A_n + B_n\cdot x_t \implies p_{n+1,t} = A_{n+1} + B_{n+1}\cdot x_t$ with:
$B_n = -1 + \phi B_{n-1} = -\frac{1-\phi^n}{1-\phi}$
$A_n = A_{n-1} + B_{n-1}(\mu - \lambda) + \frac12 B_{n-1}^2\sigma^2$

**Example: (Cox-Ingersoll-Ross, 1985)**
(1) **One State Variable:** $x_t$
(2) **Assume Log SDF:** $m_{t+1} = -x_t - \frac12\left(\frac\lambda\sigma\right)^2 x_t - \left(\frac\lambda\sigma\right)x_t^{0.5}\varepsilon_{t+1}$.
(3) **Under $\mathbb{P}$:** $x_{t+1} = \mu + \phi x_t + \sigma x_t^{0.5}\varepsilon_{t+1}$; with $\phi<1$, $\varepsilon_t \overset{\text{iid}}{\sim} N(0,1)$
(4) **Sol°:** Iterate on $p_{N,t} = \log\mathbb{E}_t[\exp(m_{t+1}+p_{N-1,t+1})|x_t]$
Use: $(m_{t+1}|x_t) = cst - \left(\frac\lambda\sigma\right)x_t^{0.5}\varepsilon_{t+1}$ and $(p_{N-1,t+1}|x_t) \perp \varepsilon_{t+1}$
$\Rightarrow (m_{t+1} + p_{N-1,t+1}|x_t) \sim N(\mathbb{E}_t[m_{t+1}+p_{N-1,t+1}], \text{SD}_t[m_{t+1}+p_{N-1,t+1}])$
$\implies p_{N,t} = \mathbb{E}_t[m_{t+1} + p_{N-1,t+1}] + \frac12\text{Var}_t[m_{t+1} + p_{N-1,t+1}]$
$\implies p_{N,t} = \mathbb{E}_t[m_{t+1} + p_{N-1,t+1}] + \frac12\text{Var}_t[m_{t+1}]$
$+\frac12\text{Var}_t[p_{N-1,t+1}] + \text{Cov}_t(m_{t+1}, p_{N-1,t+1})$
$\implies p_{1,t} = \mathbb{E}_t[m_{t+1}] + \frac12\text{Var}_t[m_{t+1}]$ therefore:
$p_{N,t} = p_{1,t} + \mathbb{E}_t[p_{N-1,t+1}] + \frac12\text{Var}_t[p_{N-1,t+1}] + \text{Cov}_t(m_{t+1}, p_{N-1,t+1})$
$\implies p_{1,t} = -x_t - \frac12\left(\frac\lambda\sigma\right)^2 x_t + \frac12\left(\frac\lambda\sigma\right)^2 x_t = -x_t$
$\implies$ **Short Rate:** $x_t = y_{1t} = \log(1+r^f) \longrightarrow$ **mean-reverting** AR(1)
(5) **Guess** $p_{n,t} = A_n + B_n\cdot x_t$, therefore:
$p_{n+1,t} = p_{1,t} + \mathbb{E}_t[p_{n,t+1}] + \frac12\text{Var}_t[p_{n,t+1}] + \text{Cov}_t(m_{t+1}, p_{n,t+1})$
$= -x_t + [A_n + B_n(\mu+\phi x_t)] + \frac12[B_n^2\sigma^2 x_t] + [-B_n\lambda x_t]$
$= \underbrace{[A_n + \mu B_n]}_{A_{n+1}} + \underbrace{[-1 + (\phi-\lambda)B_n + (1/2)\sigma^2 B_n^2]}_{B_{n+1}} x_t$

**Proposition (Bond Pricing with Real Returns)**
**Real:** SDF$=\eta_t^r$, Price$=P_t^r \longrightarrow P_{n,t}^r = \mathbb{E}_t\left[\eta_{t+1}^r P_{n-1,t+1}^r\right]$
**Nominal:** SDF$=\eta_t$, Price$=P_t$ and **Price Level:** $\Pi_t$
$\implies P_{n,t} = P_{n,t}^r\Pi_t = \mathbb{E}_t\left[\eta_{t+1}^r P_{n-1,t+1}\frac{\Pi_t}{\Pi_{t+1}}\right]$
$\eta_{t+1} = \eta_{t+1}^r\frac{\Pi_t}{\Pi_{t+1}} =: \eta_{t+1}^r\frac{1}{1+\pi_{t+1}}$
Fisher Eq: $\frac{1}{1+i_t}(1+\mathbb{E}_t[\pi_{t+1}]) = \frac{1}{1+r_t} \implies r_t \approx i_t - \mathbb{E}_t[\pi_{t+1}]$

## FTAP: Options

**Underlying Asset** $X_t$ = Payoff of asset at time $T>t$,
s.t. $f(X_T)$ = payoff of derivative security at $T>t$, $f(\cdot)$ known.
**Prop:** Derivative Price: Given $\eta$: $P_t^D=\mathbb{E}_t\left[\eta_{t:T}\cdot f(X_T)\right]$
$T=1$: **maturity/exercise date** ; $K$ = **strike/exercise price**
**Payoff:** **Call** $c_1=[S_1-K]_+$, **Put** $p_1=[K-S_1]_+$.
$c(S,K)=V(c_1)$: **call price** ; $p(S,K)=V(p_1)$: **put price**
$c(S,K)\searrow$ + convex in $K$ and $p(S,K)\nearrow$ in $K$
**(Intrinsic Value)** Call: $I=S-K$ ; Put: $I=K-S$
**In-The-Money:** $I>0$, $S>K$ (call), $K>S$ (put)
Portf of Options: $c(S'\theta,K'\theta)\le\sum_{i=1}^N\theta_i c(S_i,K_i)$
and $p(S'\theta,K'\theta)\le\sum_{i=1}^N\theta_i p(S_i,K_i)$
**Prop:** Bounds: $\left[S-\frac{K}{1+r^f}\right]_+\le c(S,K)\le S$
**Prop:** Put-Call Parity: $c(S,K)+\frac{K}{1+r^f}+D_{t=0}=p(S,K)+S$
**Price:** $C(S,K)$ **American Call** , $P(S,K)$ **American Put**
**Prop:** $D$= Divid ($t=0$), $S$=ex-divid price $P(S,D,K)$
$=\max\{K\text{-}S\text{-}D,p(S,K)\}$; $C(S,D,K)=\max\{S\text{+}D\text{-}K,c(S,K)\}$
**Prop:** Binomial $c(S,K)=\phi_u[uS-K]_++\phi_d[dS-K]_+$ with
$\begin{cases}c_u=[uS-K]_+\\ c_d=[dS-K]_+\end{cases}$ $\begin{cases}\phi_u=\frac{1}{1+r^f}\frac{1+r^f-d}{u-d}\\ \phi_d=\frac{1}{1+r^f}\frac{u-1-r^f}{u-d}\end{cases}$ $(\because)$ $\begin{cases}S=\phi_u uS+\phi_d dS\\ \frac{1}{1+r^f}=\phi_u+\phi_d\end{cases}$
Replic Portfolio $\theta=[\theta_S,\theta_B]'$ $c(S,K)=\theta_S S+\theta_B\frac{1}{1+r^f}$ with
$\begin{cases}\text{payoff}_u=\theta_S uS+\theta_B\overset{!}{=}c_u\\ \text{payoff}_d=\theta_S dS+\theta_B\overset{!}{=}c_d\end{cases}$ $(\because)$ $\begin{cases}\theta_S=\frac{c_u-c_d}{(u-d)S}\\ \theta_B=\frac{uc_d-dc_u}{u-d}\end{cases}$
Risk-Neut $c(S,K)=\frac{\mathbb{E}^{\mathbb{Q}}[[S-K]_+]}{1+r^f}=\frac{qc_u+(1-q)c_d}{1+r^f}$ $(\because)$ $q=\frac{1+r^f-d}{u-d}$

## Exact Arbitrage Pricing Theory (APT)

**Prop:** SDF Portf: $\eta=\mathbb{E}^{\mathbb{P}}[r_{-\eta}]+\mathbb{E}^{\mathbb{P}}[\eta]-r_{-\eta}$
$\mathbb{E}$[returns] Decomp $\pi_n=\mathbb{E}^{\mathbb{P}}\left[r_n-r^f\right]=\frac{\mathrm{Cov}^{\mathbb{P}}(r_n,r_{-\eta})}{\mathrm{Var}^{\mathbb{P}}(\eta)}\cdot\lambda=\beta_n\cdot\lambda$
with $\lambda:=\mathbb{E}^{\mathbb{P}}[r_{-\eta}]-r^f=(1+r^f)\mathrm{Var}^{\mathbb{P}}(\eta)$
**Idiosyncr Risk:** $r_{n,t}-r^f=\alpha_n+\beta_n(r_{-\eta,t}-r^f)+\varepsilon_{n,t}$
**Prop:** Var Deco: $\mathrm{Var}\left(r_n-r^f\right)=\beta_n^2\mathrm{Var}\left(r_{-\eta}-r^f\right)+\mathrm{Var}(\varepsilon_n)$
**Factor Structure** $F$= Basis for $D$, rank$(D)=K\le M\longrightarrow\exists$ redund
$F=[F_1,\dots,F_K]\in\mathbb{R}^{M\times K}$, and $F_k\in\mathbb{R}^{M\times1}$; **Beta** $\beta_Z\in\mathbb{R}^K$;
$C=\{D\theta:\theta\in\mathbb{R}^N\}$: $\forall Z\in C,\exists\beta_Z=[\beta_{Z1},\dots,\beta_{ZK}]'$ s.t. $Z=F\beta_Z$
Factor Pricing: $R_n=\bar R_n+\varepsilon_n=\bar R_n+F\beta_n=\bar R_n+\sum_{k=1}^K F_k\beta_{nk}$
$\Longrightarrow R=\bar R+\varepsilon=\bar R+F\beta$ (so need $\mathbb{E}^{\mathbb{P}}[F\beta]=0$, $\mathbb{E}^{\mathbb{P}}[\varepsilon_n]=0$)
$\longrightarrow$ One RF factor $\iota:=F_1=\mathbf{1}_M$ and $(K-1)$ risk factors
**Prop:** Exact APT: Let $R_n=\bar R_n+F\beta_n$, $n=1,\dots,N$, where:
(1) $F=[F_1,\dots,F_K]$ ($K$ risk factors, $\mathbb{E}^{\mathbb{P}}[F_k]=0$)
(2) $\beta_n=[\beta_{n1},\dots,\beta_{nK}]'$ (asset $n$'s beta)
N.A. $\Longrightarrow$ $\bar R_n-R^f=\bar r_n-r^f=\sum_{k=1}^K\lambda_k\beta_{nk}=\lambda'\beta_n$ , $n=1,\dots,N$
where $\lambda_k=-\mathbb{E}^{\mathbb{Q}}[F_k]=\bar r_k-r^f$ ($\bar r_k$=fact $k$) and $\lambda=[\lambda_1,\dots,\lambda_K]'$
**Risk-Free Portf:** set $\beta_{\text{portf}}=\sum_k\theta_k\beta_k=0$, $\sum_k\theta_k=1$
**Factor Portf:** set $\beta_{\text{portf}}=\sum_k\theta_k\beta_k=1$, $\sum_k\theta_k=1$

## General Arbitrage Pricing Theory (APT)

Model for Ret: $r_n=\bar r_n+\sum_{k=1}^K\beta_{nk}F_k+\varepsilon_n$, for $(r=\bar r+F\beta+\varepsilon)$
with: **(1)** $\mathbb{E}^{\mathbb{P}}[F_k]=\mathbb{E}^{\mathbb{P}}[\varepsilon_n]=\mathbb{E}^{\mathbb{P}}[\varepsilon_n|F_k]=0$, $\forall k,n$
**(2)** $\mathbb{E}^{\mathbb{P}}\left[\varepsilon_n^2\right]=\sigma_n^2<v<\infty$, and $\mathbb{E}^{\mathbb{P}}[\varepsilon_n\varepsilon_{n'}]=0\ \forall n\neq n'$
$\mathrm{Var}^{\mathbb{P}}(r_n)=\beta_n'\mathbb{E}^{\mathbb{P}}[F'F]\beta_n+\mathrm{Var}^{\mathbb{P}}(\varepsilon_n)$, $\mathrm{Cov}^{\mathbb{P}}(r_i,r_j)=\beta_i'\mathbb{E}^{\mathbb{P}}[F'F]\beta_j$
**Portf Ret:** $r_\theta=\bar r_\theta+F\beta_\theta+\varepsilon_\theta$ ($r_\theta=\theta r$, $\bar r_\theta=\theta\bar r$, $\beta_\theta=\theta\beta$, $\varepsilon_\theta=\theta\varepsilon$)
**Div Port:** $\theta_n=O(1/n)$ , $\theta=[\theta_1,\dots,\theta_N]'$, $\theta'\mathbf{1}_N=\sum_{n=1}^N\theta_n=1$
**Div Seq:** $\{\theta_n\}_{n=1}^\infty$, with $\theta_n'\mathbf{1}_n=\sum_{i=1}^n\theta_{n,i}=1$, is Well-Diversified
$\iff\exists\kappa\in(0,\infty)$, s.t. $\theta_{n,i}^2<\kappa/n^2$ , $\forall i=1,\dots,n$, $\forall n\ge1$
Div Thm: $\{\theta_n\}_{n=1}^\infty$ div seq: $\mathrm{Var}^{\mathbb{P}}(\varepsilon_{\theta_n})=\mathrm{Var}^{\mathbb{P}}\left(\sum_{i=1}^n\theta_{n,i}\varepsilon_i\right)\to0$
**Asy Arb:** $\{\theta_n\}_{n=1}^\infty$ s.t. $\mathbf{1}_n'\theta_n=0$, $\mathbb{E}^{\mathbb{P}}[r_{\theta_n}]\to\alpha>0$, $\mathrm{Var}^{\mathbb{P}}(r_{\theta_n})\to0$
**Prop:** NAA $\Longrightarrow$ NA

**Prop:** APT: Given NAA + $K$-Fact: $\exists r^f\in\mathbb{R},\lambda=[\lambda_1,\dots,\lambda_K]'\in\mathbb{R}^K$:
$\sum_{i=1}^n\left[\bar r_i-(r^f+\lambda'\beta_i)\right]^2=\sum_{i=1}^n\left[\bar r_i-\left(r^f+\sum_{k=1}^K\lambda_k\beta_{ik}\right)\right]^2<A<\infty$
NAA $\Longrightarrow$ approx. factor pricing: $\bar r_i-r^f\approx\sum_{k=1}^K\lambda_k\beta_{ik}$, $\forall i$

# Optimal Portfolio Choices

## Expected Utility (E.U.) Theory

**Continuity:** $\forall c\in C$: $\{a\in C:a\succcurlyeq c\}$ & $\{b\in C:b\preccurlyeq c\}$ closed
$\iff\forall\{a_n\}\to a,\{b_n\}\to b\in C:a_n\succcurlyeq b_n\Rightarrow a\succcurlyeq b$
**Insatiability:** $a>b\Longrightarrow a\succcurlyeq b$ (more $\succcurlyeq$ less)
**Convexity:** $\forall\alpha\in(0,1)$ : $a\succcurlyeq b$ & $c\succcurlyeq b\Longrightarrow\alpha a+(1-\alpha)c\succcurlyeq b$
**Prop:** Convex $\succcurlyeq\ \Longrightarrow$ convex sets of preferred bundles $\{a\in C:a\succcurlyeq c\}$
**EU:** over consump path/lottery $u(c,p)=\sum_{\omega\in\Omega}p_\omega u_\omega(c_0,c_{1\omega})$, $c\in C$.
**Continuity:** $\forall$ consumption $c\in C$, probas $p_a,p_b,p_c$:
$[c,p_a]\succcurlyeq[c,p_b]\succcurlyeq[c,p_c]\Longrightarrow\exists\alpha\in(0,1):[c,p_b]\sim[c,(1-\alpha)p_a+\alpha p_c]$
**Indep:** $\forall$ consumption $c\in C$, $\alpha,p_a,p_b,p_c\in(0,1)$: $[c,p_a]\succcurlyeq[c,p_b]$
$\Longrightarrow[c,(1-\alpha)p_a+\alpha p_c]\succcurlyeq[c,(1-\alpha)p_b+\alpha p_c]$
**Assume:** $u(c,p)=u(c_0)+\rho\sum_{\omega\in\Omega}p_\omega u(c_{1\omega})$ with $\rho\in(0,1)$
**Marginal Utility:** At consumption level $c$: $u'(c)$
**Prop:** Insatiability $\Longrightarrow$ $u$ is strictly $\nearrow$, $u'>0$
**Concave:** $u(\alpha x+(1-\alpha)x')\ge\alpha u(x)+(1-\alpha)u(x')$
**Prop:** $u$ concave & twice differentiable $\iff$ $u''\le0$, $u'\searrow$

## Risk-Aversion

**Risk Aversion:** $\mathbb{E}[u(w+x)]\le\mathbb{E}[u(w)]$ for any $\mathbb{E}[x]=0$.
**Risk Prem:** $\mathbb{E}[x]=0$, EU $u$, wealth $w$: $\mathbb{E}[u(w+x)]=u(w-\pi)$
**Certainty Equivalent:** $u(w_{CE})=\mathbb{E}[u(w)]$
**Abs RA:** $A(w)=-\frac{u''(w)}{u'(w)}$ **Prop:** Small Gamble $x$: $\pi\approx\frac12A\cdot\mathrm{Var}(x)$
$(\because)$ $\mathbb{E}[u(w+x)]=u(w)+\frac12u''(w)\mathbb{E}\left[x^2\right]\overset{!}{=}u(w-\pi)=u(w)\text{-}u'(w)\pi$
**Rel RA:** $R(w)=-w\frac{u''(w)}{u'(w)}$ **Prop:** Small Risk $wx$: $\pi_R\approx\frac12R\cdot\mathrm{Var}(x)$
$(\because)$ $\mathbb{E}[u(w(1+x))]=u(w(1-\pi_R))$
**Prop:** Pratt: Agents 1 & 2 w/ EU $u_1$ & $u_2$: $A_1(w)\ge A_2(w)\ \forall w$
$\iff u_1(u_2^{-1}(\cdot))$ concave $\iff\pi_1\ge\pi_2$, $\forall w$ & fair gambles $x$
$\iff\exists f$ s.t. $f'>0,f''\le0$ & $u_1(w)=f(u_2(w))$
**Risk Neutr:** $A(w)=R(w)=0$, **Linear EU:** $u(w)=w$
**CARA:** $A'(w)=0$, **Negative Exp EU:** $u(w)=-e^{-aw}$, $a>0$
$\Longrightarrow$ **CARA** agents: $A(w)=a$, $R(w)=aw$
**CRRA** $R'(w)=0$, **Power EU** $u(w)=\frac{1}{1-\gamma}w^{1-\gamma}$, $\gamma>1$
$\Longrightarrow$ **CRRA** agents: $A(w)=\gamma/w$, $R(w)=\gamma$ **Prop:** $\gamma\to1\Rightarrow$ Log EU
**Log EU:** $u(w)=\log w\Longrightarrow$ **CRRA** agents: $A(w)=1/w$, $R(w)=1$
**IARA/DARA:** $A'(w)\gtrless0$ ; **IRRA/DRRA:** $R'(w)\gtrless0$
**Quadratic EU:** $u(w)=w-0.5aw^2$, $a>0$, $w\in[0,1/a]$
$\Longrightarrow$ **IARA** agents: $A(w)=\frac{a}{1-aw}$, $R(w)=\frac{aw}{1-aw}$

## Optimal Consumption/Portfolio Choice

**Setting:** $N$ non-redund assets, payoff $D$, price $P$
Agent: Endowt $e=[e_0,e_1']'$, Consump plan $c=[c_0,c_1']'$, Portf $\theta$
EU: $u(c)=u_0(c_0)+\mathbb{E}[u_1(c_1)]$ with $u_t'>0$, $u_t''<0$ ($t=0,1$)
Agnt Opt: $\max_\theta u_0(c_0)+\mathbb{E}[u_1(c_1)]$ s.t. $c_0=e_0-P'\theta$, $c_1=e_1+D\theta$
$P'\theta=e_0-c_0$=time-0 savings
**Assumptions** **(Complete Market)**
Complete set of AD securities, State Price $\phi\gg0$
Agent: endowement $e=[e_0,e_1']'$, wealth $w=e_0+\phi'e_1$
Budget: $B(e)=\{c:c_0+\phi'c_1=w\}$ (Simplify: ignore $c\ge0$)
Marginal cost $=\phi_\omega$: Additional \$1 in asset $\omega\Longrightarrow c_{1\omega}\nearrow$ by $1/\phi_\omega$
**Proposition (Optimization)** $\max_{c_0+\phi'c_1=w}u_0(c_0)+\sum_\omega p_\omega u_1(c_{1\omega})$
**Lagrang:** $\mathcal{L}=u_0(c_0)+\sum_\omega p_\omega u_1(c_{1\omega})-\lambda\left[c_0+\phi'c_1-w\right]\to\partial_{c_0},\partial_{c_1}$
**FOC:** $\lambda=u_0'(c_0)\longrightarrow$ marginal value of wealth
$\lambda\phi_\omega=p_\omega u_1'(c_{1\omega})=\frac{\partial\mathbb{E}[u_1(c_1)]}{\partial\theta_\omega}\longrightarrow$ margin benefit of $\nearrow c_{1\omega}=\theta_\omega D_{1\omega}$
$\eta_\omega=\frac{\phi_\omega}{p_\omega}=\frac{u_1'(c_{1\omega})}{u_0'(c_0)}$= intertemp marg rate of substitution, $\frac{\phi_\omega}{\phi_{\omega'}}=\frac{p_\omega u_1'(c_{1\omega})}{p_{\omega'}u_1'(c_{1\omega'})}$
**Prop:** $u_t$ strictly concave $\Longrightarrow$ $u_t'$ strictly $\searrow$ & $u_t'^{-1}$ exists

**Theorem (Optimal Portfolio Choice***)** Solve FOC:
$c_0=u_0'^{-1}(\lambda)$ and $c_{1\omega}=u_1'^{-1}\left(\lambda\frac{\phi_\omega}{p_\omega}\right)\ \forall\omega\in\Omega$
where $\lambda$ solves budget constraint: $w=e_0+\phi'e_1=c_0(\lambda)+\phi'c_1(\lambda)$
**Characterization of Optimal Portfolio:**
**Proposition (Optimization)** $\max_\theta u_0(e_0-P'\theta)+\mathbb{E}[u_1(e_1+D\theta)]$
$=\max_\theta u_0(e_0-P'\theta)+\sum_\omega p_\omega u_1\left(e_{1\omega}+\sum_{n=1}^N\theta_nD_{1\omega n}\right)$
**Euler Eqn:** FOC: $u_0'(c_0)P_n=\mathbb{E}[u_1'(c_1)D_n]$, $n=1..N$
**Prop:** Portf Decomp: Agent's $t=0$ savings: $w=e_0-c_0=P'\theta$
$\Longrightarrow$ Optimal consump/portf choice: $\max_w\{u_0(e_0-w)+v_1(w)\}$
$v$ **function:** $v_1(w)=\max_{\{\theta:P'\theta=w\}}\mathbb{E}[u_1(e_1+D\theta)]$
**Example: (Special Case)** $e_1=0$ (agent endowed only with $e_0$ cash)
$\Longrightarrow$ Portf Choice Pb: $v(w)=\max_{\{\theta:P'\theta=w\}}\mathbb{E}[u_1(D\theta)]$
Riskless asset: asset $N$ with gross return $R_n=1+r^f$
$a_n=\theta_nP_n$: \$ invested in asset $n\Longrightarrow w=\sum_n a_n$ **total investment**
Portf payoff: $\tilde w=D\theta=\sum_{n=1}^N a_nR_n=w(1+r^f)+\sum_{n=1}^{N-1}a_n(r_n-r^f)$
**Excess Return** of asset $n$: $r_n-r^f$
General Pb: $r=[r_1,\dots,r_{N-1}]'$ risky; $a=[a_1,\dots,a_{N-1}]'$ investment;
Optimal Portf: $\max_a\mathbb{E}[u(\tilde w)]=\max_a\mathbb{E}\left[u\left(w(1+r^f)+(r-r^f\iota')a\right)\right]$
$\Longrightarrow$ FOC: $\mathbb{E}\left[u'(\tilde w)(r_n-r^f)\right]=0\ \forall n=1..N-1$
**Properties of Optimal Portfolio:**
**Case 1:** Assume only **ONE** risky asset:
**Prop:** $\tilde w=w(1+r^f)+a(r-r^f)$ $(\because)$ borrow at $r^f$, invest risky
**Prop:** Opt Investment $a$: Agent = strictly RA
$a>0\iff\bar r>r^f$ ; $a<0\iff\bar r<r^f$ ; $a=0\iff r=r^f$
**Prop:** risk-premium$>0\Longrightarrow$ agent invest at least $\varepsilon$ in risky asset
**Proposition (Abs RA)** Assume $\bar r-r^f>0$ (so $a>0$)
$a'(w)>0\Leftrightarrow A'(w)<0$ (DARA) ; $a'(w)=0\Leftrightarrow A'(w)=0$ (CARA)
$a'(w)<0\Leftrightarrow A'(w)>0$ (IARA $\longrightarrow$ very rare)
**Prop:** You see from FOC diff: $\frac{da}{dw}=-(1+r^f)\frac{\mathbb{E}\left[u''(\tilde w)(r-r^f)\right]}{\mathbb{E}\left[u''(\tilde w)(r-r^f)^2\right]}$
**Relative Propensity:** for investor in risky asset: $e(w)=\frac{w}{a}\frac{da}{dw}$
**Note:** $e(w)=1\Leftrightarrow a(w)=\bar a\cdot w$: risky invest$^M$ = CST fract° of wealth
**Prop:** Rel RA: Assume $\bar r-r^f>0$ (so $a>0$)
$e(w)>1\Leftrightarrow R'(w)<0$ (DRRA) ; $e(w)=1\Leftrightarrow R'(w)=0$ (CRRA)
$e(w)<1\Leftrightarrow R'(w)>0$ (IRRA $\longrightarrow$ very rare)
**Case 2:** Assume **MULTIPLE** risky assets:
**Prop:** $\tilde w=w[(1+r^f)+a'(r-r^f\iota)]$
**Theorem (Opt Investment $a$)** $a=0\Leftrightarrow\mathbb{E}[r_n]=r^f\ \forall n=1..N-1$
**Theorem (Opt Investment $a$ II)** Some risk-prem on risky assets$\neq0$
$\Longrightarrow\mathbb{E}[r_{\text{portf}}]>r^f$ (i.e., $\sum_{n=1}^{N-1}a_n(\mathbb{E}[r_n]-r^f)\ge0$)
**Prop:** USEFUL FORMULAS: $\tilde w=w(1+r^f)+a(r-r^f)$
$\tilde w=w[(1+r^f)+a'(r-r^f\iota)]=w[R^f+a'(\tilde R-R^f\iota)]$
$c_1=(w_0-c_0)R^f+a(\tilde R-R^f)$ ; $w_0=e_0-c_0$
$c_1=e_1+w(1+r^f)+a(r-r^f)=e_1+(e_0-c_0)(1+r^f)+a(r-r^f)$
$\tilde w=\sum_{n=1}^N\underbrace{\theta_nD_n}_{\text{Asset }n\text{ Payoff}}+\frac{1}{P_0}\left(\sum_{n=1}^N\underbrace{P_n}_{\text{t=0 wealth (endowment)}}-\sum_{n=1}^N\underbrace{\theta_nP_n}_{\text{wealth invested in risky assets}}\right)$
**FSD:** $A\gtrsim_{\text{FSD}}B\iff\forall u'\ge0:\mathbb{E}[u(r_A)]\ge\mathbb{E}[u(r_B)]$
**Prop:** $A\gtrsim_{\text{FSD}}B\Longrightarrow\bar r_A\ge\bar r_B$ but converse FALSE!
**Prop:** $A\gtrsim_{\text{FSD}}B\Leftrightarrow F_A(x)\le F_B(x)\ \forall x\Leftrightarrow r_A\overset{d}{\sim}r_B+\varepsilon$, with $\varepsilon\ge0$
Ordering: $A\gtrsim_{\text{FSD}}B\Longrightarrow$: for $u'>0,u''<0$
$\max_a\mathbb{E}\left[u(w(1+r^f)+a(r_A-r^f))\right]\ge\max_a\mathbb{E}\left[u(w(1+r^f)+a(r_B-r^f))\right]$
**SSD:** $A\gtrsim_{\text{SSD}}B\iff\forall u''\le0:\mathbb{E}[u(R_A)]\ge\mathbb{E}[u(R_B)]$
**Prop:** ONLY WORKS IF $\bar R_A=\bar R_B$ !
**Prop:** $A\gtrsim_{\text{SSD}}B\Longrightarrow\mathrm{Var}(R_A)\le\mathrm{Var}(R_B)$ but converse FALSE!
**Prop:** Rothschild-Stiglitz: $A\gtrsim_{\text{SSD}}B$
$\iff\mathbb{E}[R_A]=\mathbb{E}[R_B]$ and $\int_0^y[F_A(x)-F_B(x)]dx=:S(y)\le0\ \forall y$
$\iff R_A\overset{d}{\sim}R_B+\varepsilon$, with $\mathbb{E}[\varepsilon|R_B]=0$
**Prop:** $R_A\sim N(\mu,\sigma_A^2)$, $R_B\sim N(\mu,\sigma_B^2)$: $\sigma_A<\sigma_B\Longrightarrow A\gtrsim_{\text{SSD}}B$

## In Practice

### Consumption Choice Problem 1:

**Setup:** $M$=2 states, $N$=2 assets (1 RF + 1 risky)

$t$=1 returns: $R=\begin{bmatrix}1 & u\\ 1 & d\end{bmatrix}$, probs: $\pi_u, \pi_d$.

Agent: initial wealth $w_0$, final $w_1 = 0$.

Portf Weights (RF/Risky asset): $\alpha = [\alpha_1, \alpha_2]'$

Max Prob: $\max\limits_{c_0, c_1, \alpha} \log c_0 + \beta \mathbb{E}\left[\log c_1\right]$

2 Budget Constraints: $c_1 = (w_0 - c_0)R\alpha$

**(a) Find optimal $\alpha = \alpha(\mathbf{w_0}, \mathbf{c_0}, \mathbf{c_1})$:**

Complete Market: $\exists R^{-1} \implies \alpha = \frac{1}{w_0 - c_0} R^{-1} \cdot c_1$

**(b) Rewrite Constraints Using Only $(\mathbf{w_0}, \mathbf{c_0}, \mathbf{c_1})$:**

$\phi = P' R^{-1}$ with $P = [1, 1]'$ so $\eta_u = \frac{\phi_u}{\pi_u}, \eta_d = \frac{\phi_d}{\pi_d}$

$\implies$ Constraint: $w_0 := c_0 + \mathbb{E}\left[\eta c_1\right] = c_0 + \pi_u \eta_u c_{1,u} + \pi_d \eta_d c_{1,d}$

**(c) Optimize over $(\mathbf{c_0}, \mathbf{c_1})$ + Find $\mathbf{c_0}(\lambda), \mathbf{c_1}(\lambda)$:**

$\max\limits_{c_0, c_1} \log c_0 + \beta \mathbb{E}\left[\log c_1\right]$ s.t. $w_0 = c_0 + \pi_u \eta_u c_{1,u} + \pi_d \eta_d c_{1,d}$

$\mathcal{L} = \log c_0 + \beta \mathbb{E}\left[\log c_1\right] - \lambda\left(c_0 + \pi_u \eta_u c_{1,u} + \pi_d \eta_d c_{1,d} - w_0\right)$

$\overset{c_0}{\implies} \lambda = 1/c_0$ ; $\overset{c_{1,s}}{\implies} \lambda \pi_s \eta_s = \beta \pi_s / c_{1,s}$ with $s = u, d$

$\implies c_0 = \frac{1}{\lambda} \qquad c_{1,u} = \frac{\beta}{\lambda \eta_u} = \frac{c_0 \beta}{\eta_u} \qquad c_{1,d} = \frac{\beta}{\lambda \eta_d} = \frac{c_0 \beta}{\eta_d}$

**(d) Plug $\lambda$ in constraint + Get $\lambda = \lambda(\mathbf{w_0})$:**

$w_0 = c_0 + \pi_u \eta_u c_{1,u} + \pi_d \eta_d c_{1,d} = \frac{1}{\lambda} + \frac{\pi_u \beta}{\lambda} + \frac{\pi_d \beta}{\lambda}$

$\implies \lambda = \frac{1+\beta}{w_0}$

**(e) Get $\mathbf{c_0}(\mathbf{w_0}), \mathbf{c_1}(\mathbf{w_0})$:**

$c_0 = \frac{w_0}{1+\beta} \qquad c_{1,u} = \frac{\beta}{1+\beta}\frac{w_0}{\eta_u} \qquad c_{1,d} = \frac{\beta}{1+\beta}\frac{w_0}{\eta_d}$

### Consumption Choice Problem 2:

**Setup:** 1 RF (return $r^f$) + 1 Risky asset:

$$r = \bar{r} + \sigma\varepsilon \quad \varepsilon \sim N(0,1) \quad (\bar{r} > r^f)$$

Agent: $e_0 > 0$ and $e_1 = h\varepsilon$ ($h > 0$)

Maximize: $\max\limits_{c_0, c_1} -e^{-\alpha c_0} - \rho \mathbb{E}\left[e^{-\alpha c_1}\right]$, $\alpha > 0$ cst

**(a) Invest $a$ in risky asset:**

**Write $\mathbf{t = 1}$ consumption $\mathbf{c_1 = c_1(e_0, e_1, c_0, a, r^f, r)}$**

$c_1 = e_1 + w(1 + r^f) + a(r - r^f) = e_1 + (e_0 - c_0)(1 + r^f) + a(r - r^f)$

**(b) Write optimal portf choice problem:**

$\max\limits_{c_0, a} -e^{-\alpha c_0} - \rho \mathbb{E}\left[e^{-\alpha\left(e_1 + (e_0 - c_0)(1 + r^f) + a(r - r^f)\right)}\right]$

**(c) Write FOC:** $\frac{d}{da}$ and $\frac{d}{dc_0}$

**(d) Solve FOC for Opt Portf Choice problem:** Get $a, c_0$

**(e) How does h influences $\mathbf{c_0}$ and a?**

$c_0 \searrow$ in $h$. Higher uncertainty about $e_1 \implies$ lower certainty equivalent of this payoff. When $h$ is high: agent feels poorer $\Rightarrow$ wants to consume less.

$a \searrow$ in $h$. Higher uncertainty about $e_1 \implies$ less willingness to invest in the risky asset (adds risk to $c_1$).

### Consumption Choice Problem 3:

**Setup:** 1 RF (asset 0) $+N$ risky assets: $n = 1, .., N$

Returns: $R_0 = R^f = 1$ and $D_n = \bar{D} + \varepsilon_n$ $(n = 1..N)$, $\varepsilon_n \overset{\text{iid}}{\sim} N(0, \sigma)$

Prices: $P_0 = 1$, $P_n = P$ for all $n$

Agent: CARA $u(w) = -e^{-aw}$, $a > 0$

Endowment: 1 share of each asset $n$, 0 shares of RF.

Portfolio holdings of risky assets: $\theta = [\theta_1, \ldots, \theta_N]' \in \mathbb{R}^N$

**(a) Write agent's wealth $\tilde{\mathbf{w}}$ at $\mathbf{t = 1}$:**

$$\tilde{w} = \sum_{n=1}^N \underbrace{\theta_n D_n}_{\substack{\text{Asset } n\\ \text{Payoff}}} + \frac{1}{P_0}\Bigg(\sum_{n=1}^N \underbrace{P_n}_{\substack{\text{t=0 wealth}\\ \text{(endowment)}}} - \sum_{n=1}^N \underbrace{\theta_n P_n}_{\substack{\text{wealth invested}\\ \text{in risky assets}}}\Bigg)$$

**(b) Write Optimal Portf Choice Pb:**

$\max\limits_{\theta} \mathbb{E}\left[-e^{-a\tilde{w}}\right]$ s.t. $\tilde{w} = \sum_{n=1}^N \theta_n D_n + \sum_{n=1}^N (1 - \theta_n) P_n$

$D_n \overset{\text{iid}}{\sim} N(\bar{D}, \sigma^2)$

$\implies \tilde{w} \sim N\left(\sum_{n=1}^N \theta_n \bar{D}_n + \sum_{n=1}^N (1 - \theta_n) P_n, \sigma^2 \sum_{n=1}^N \theta_n^2\right)$

$\Rightarrow -a\tilde{w} \sim N\left(-a\sum_{n=1}^N \theta_n \bar{D}_n - a\sum_{n=1}^N (1 - \theta_n) P_n, a^2\sigma^2 \sum_{n=1}^N \theta_n^2\right)$

$\mathbb{E}\left[-e^{-a\tilde{w}}\right] = -\exp\left(-a\sum_{n=1}^N \theta_n \bar{D}_n - a\sum_{n=1}^N (1 - \theta_n) P_n + \frac{a^2}{2}\sigma^2 \sum_{n=1}^N \theta_n^2\right)$

$$\implies \max_{\theta} \sum_{n=1}^N \theta_n \bar{D}_n + \sum_{n=1}^N (1 - \theta_n) P_n - \frac{a}{2}\sigma^2 \sum_{n=1}^N \theta_n^2$$

**(c) Solve Optimal Portf Pb:** FOC w.r.t. $\theta_n$

$\bar{D} - P_n - a\sigma^2\theta_n = 0 \implies \theta_n = \frac{\bar{D} - P_n}{a\sigma^2}$

**(d) Show: for different values of RA a, 2-Fund Separation Holds:**

Initial Wealth of Agents: $w_0 = \sum_{n=1}^N P_n$

$\implies$ agents invest optimally fractions

$\frac{\theta_n P_n}{w_0} = \frac{1}{a}\frac{(\bar{D} - P_n)P_n}{w_0\sigma^2}$ of wealth in asset $n$

and $1 - \frac{1}{a}\frac{(\bar{D} - P_n)P_n}{w_0\sigma^2}$ in RF asset

$\implies$ agents hold lin comb of RF asset

and risky portf $x_M = [x_1, \ldots, x_N]'$ ($x_n = \frac{(\bar{D} - P_n)P_n}{w_0\sigma^2}$)

Depending on RA: Hold $\frac{1}{a}X_M$ in risky & $1 - \frac{1}{a}X_M'\iota$ in RF

**(e) If agent = only agent in market**

**Find Equilibrium Risky Prices $\mathbf{P_n}$**

Market Clearing: $1 \overset{!}{=} \theta_n = \frac{\bar{D} - P_n}{a\sigma^2} \implies P_n = \bar{D} - a\sigma^2$

**(f) Find Risk Premium on Risky Assets + $\mathbf{N} \to \infty$ Limit:**

$\pi_n = \bar{R}_n - R^f = \mathbb{E}\left[R_n\right] - 1 = \frac{\mathbb{E}[D_n]}{P_n} - 1 = \frac{a\sigma^2}{\bar{D} - a\sigma^2}$ indep of $N$

**(g) Does APT Hold in this Market when $\mathbf{N} \to \infty$?**

$\exists$ Asymptotic Arbitrage in this Market: (so APT can't hold)

Seq of arb portfs: $\theta_0^N = -1$, $\theta_n^N = \frac{1}{N}\ \forall n$

$\implies \mathbb{E}\left[r_{\theta N}\right] = \sum_{n=1}^N \frac{1}{N}\bar{r}_n - 1 = \bar{r}_1 - 1 = \pi_1 = \frac{a\sigma^2}{\bar{D} - a\sigma^2} > 0$

While $\text{Var}\left(r_{\theta N}\right) = \sum_{n=1}^N \frac{1}{N^2}\text{Var}\left(\varepsilon_n\right) = \frac{\sigma^2}{N} \to 0$